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Study of Fibonacci and Lucas Sequences, With the Help of Hyperbolic Functions

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Abstract

In a series of five papers we presented some fundamental knowledge about hyperbolic functions and 184 properties expressed in 193 equalities. After in a previous paper, using hyperbolic functions, we showed that there is a field of real functions of real variable isomorphic with the field of real numbers ($\boldsymbol{R}$,+,·), in this paper we will present a study of Fibonacci and Lucas sequences, with the help of these functions. Thus, starting from the definition / recurrence relations of the Fibonacci and Lucas sequences, we will list here 18 relations between the hyperbolic functions and these sequences, followed by 12 consequences of them, i.e. 12 relations between the terms of the Fibonacci and Lucas sequences. All these relationships will be fully proven; moreover, some will be proven in two (or even three!) ways.

Keywords: Fibonacci sequence, Lucas sequence, hyperbolic functions

1. Introduction

As part of a larger project entitled "*Training and developing the competences of children, students and teachers to solve problems / exercises in Mathematics*", in five recent papers with the same generic name as this one and numbered with (I), (II), (III), respectively (IV), I presented the definitions, the consequences immediate resulting from these and a series of 38 properties of hyperbolic functions, properties that we divided into four groups, as follows: A) "*Trigonometric*" properties - nine properties; B) The derivatives of hyperbolic functions - six properties; C) The primitives (indefinite integrals) of hyperbolic functions – six properties and D) The monotony and the invertibility of hyperbolic functions - 17 properties. That in paper (I). In paper (II) I continued this approach and I presented another 54 properties of these functions, properties that have divided into three groups, as follows: E) Other properties "*trigonometric*" - 42 properties; F) Immediate properties of the inverse of hyperbolic functions – six properties and G) The derivatives of the inverse of hyperbolic functions - six properties. In paper (III) also I continued this approach and I presented another 36 properties of these functions, properties that we will divide into three groups, as follows: H) Properties „*integral*" and rewithrrence formulas - 11 properties; I) Relations between the inverse of hyperbolic functions - five properties and J) Relations between the hyperbolic functions and the inverses of other hyperbolic functions - 20 properties. In paper (IV) continuing those presented in the first three papers I presented and proved another 32 properties of these functions, properties that properties that we will classify them in the same group K): Sums and differences of inverse hyperbolic functions. Finally, in paper (V) I finished the presentation of the properties of these functions with two more categories: L) - Other relationships between the inverses of hyperbolic functions (four properties) and M) - Taylor series expansion of hyperbolic functions and their inverses (12 properties).

Because there are a total of 184 properties, expressed in 193 equalities, which we cannot repeat here, and because we use some of them, we will present their numbering, as they appear in the works mentioned above. Thus, the attentive and interested reader of these issues, to understand very well the proofs of these relationships, you will need to consult the papers where the properties of the hyperbolic functions used in this paper appear and are proven.

Therefore, this paper is also a continuation of the five papers mentioned above. The numbering just mentioned is as follows: the results numbered with (2.1) to (2.16), respectively (3.1) to (3.30) are from paper (I) - i.e. (Vălcan, 2016), those numbered with (4.1) to (4.54′) are from paper (II) – i.e. (Vălcan, 2019), those numbered with (5.1) to (5.40′) are from paper (III) – i.e. (Vălcan, (1), 2020). Results numbered with (6.1) to (6.38) are those presented and proven in paper (IV) – i.e. (Vălcan, (2), 2020) and, finally, those numbered with (7.1) to (7.23) are those presented and proven in paper (V) - that is (Vălcan, 2021).

As I mentioned in previous papers, these properties, can be used in various applications in Algebra or

Mathematical Analysis.

In (Vălcan, 2019), using the hyperbolic functions sh and ch and some of their properties, we showed that there is a field of real functions of a real variable isomorphic with the field of real numbers ($\mathbf{R}$,+,·).

In this paper we will present another application of hyperbolic functions and of some of their properties in Algebra. More precisely, here, starting from the definition / recurrence relations of two known sequences (Fibonacci and Lucas), we will present and prove 18 relations between hyperbolic functions and these sequences, followed by 12 consequences of them, i.e. 12 relations between the terms of the Fibonacci and Lucas sequences.

All these relationships will be fully proven; moreover, some will be proven in two (or even three!) ways.

Of course, the Fiboncci and Lucas strings are very well known and the literature of the field has many relationships, properties, but also their applications. In this regard, we invite the reader, attentive and interested in these issues, to consult this literature, especially the one presented in

https://mathworld.wolfram.com/FibonacciNumber.html

and

https://mathworld.wolfram.com/LucasNumber.html.

2. About the Fibonacci and Lucas sequences

Consider the sequences $(F_n)_{n\geq 1}$ and $(L_n)_{n\geq 1}$, defined as follows:

$F_1=1$, $F_2=1$

and, for every $n\in\mathbf{N}$, $n\geq 3$,

(1) $F_n=F_{n-1}+F_{n-2}$;

respectively:

$L_1=1$, $L_2=3$

and, for every $n\in\mathbf{N}$, $n\geq 3$,

(2) $L_n=L_{n-1}+L_{n-2}$.

The sequences $(F_n)_{n\geq 1}$ and $(L_n)_{n\geq 1}$, defined above are called *the Fibonacci sequence* and *the Lucas sequence*, respectively.

We notice that the two sequences have the same first term and the same recurrence relation. Because they do not have the same second term, the sequences differ; here are some values of these sequences:

n	F_n	L_n	n	F_n	L_n	n	F_n	L_n	n	F_n	L_n
1	1	1	**4**	3	7	**7**	13	29	**10**	55	123
2	1	3	**5**	5	11	**8**	21	47	**11**	89	199
3	2	4	**6**	8	18	**9**	34	76	**12**	144	322

Let us determine the general terms of the sequences $(F_n)_{n\geq 1}$ and $(L_n)_{n\geq 1}$. According to the recurrence relations, (1), respectively (2), for every $n\in\mathbf{N}^*$,

(3) $F_n=\alpha\cdot a^n+\beta\cdot b^n$

and

(4) $L_n=\gamma\cdot a^n+\delta\cdot b^n$,

where a and b are the roots of the equation:

(5) $x^2=x+1$,

and the coefficients α, β, γ, $\delta\in\mathbf{R}$, are determined from the initial conditions. Solving the equation (5), we obtain that:

$$a=\frac{1+\sqrt{5}}{2} \quad \text{and} \quad b=\frac{1-\sqrt{5}}{2}.$$

From the initial conditions for the sequence $(F_n)_{n\geq 1}$ and from the equality (3), for the first two values of n, we obtain the system:

(6) $\begin{cases} \alpha \cdot a + \beta \cdot b = 1 \\ \alpha \cdot a^2 + \beta \cdot b^2 = 1 \end{cases}$,

that is:

(7) $\begin{cases} \alpha \cdot \dfrac{1+\sqrt{5}}{2} + \beta \cdot \dfrac{1-\sqrt{5}}{2} = 1 \\ \alpha \cdot \dfrac{3+\sqrt{5}}{2} + \beta \cdot \dfrac{3-\sqrt{5}}{2} = 1 \end{cases}$.

Solving the system (7) we obtain that:

$\alpha = \dfrac{1}{\sqrt{5}}$ and $\beta = -\dfrac{1}{\sqrt{5}}$.

Now, relation (3) shows us that for every n $\in \mathbf{N}^*$,

(8) $F_n = \dfrac{1}{\sqrt{5}} \cdot \left[\left(\dfrac{1+\sqrt{5}}{2} \right)^n - \left(\dfrac{1-\sqrt{5}}{2} \right)^n \right]$

Because,

(9) $\dfrac{1}{\sqrt{5}} = \dfrac{1}{a-b}$,

From equalities (8) and (9), it follows that:

(10) $F_n = \dfrac{a^n - b^n}{a-b}$.

We return now to the sequence Lucas. From the initial conditions for the sequence $(L_n)_{n\geq1}$ and from the equality (4), for the first two values of n, we obtain the system:

(11) $\begin{cases} \gamma \cdot a + \delta \cdot b = 1 \\ \gamma \cdot a^2 + \delta \cdot b^2 = 3 \end{cases}$,

that is:

(12) $\begin{cases} \gamma \cdot \dfrac{1+\sqrt{5}}{2} + \delta \cdot \dfrac{1-\sqrt{5}}{2} = 1 \\ \gamma \cdot \dfrac{3+\sqrt{5}}{2} + \delta \cdot \dfrac{3-\sqrt{5}}{2} = 3 \end{cases}$.

Solving the system (12) we obtain that:

$\gamma=1$ and $\delta=1$.

Now, relation (4) shows us that, for every n$\in \mathbf{N}^*$,

(13) $L_n = \left(\dfrac{1+\sqrt{5}}{2} \right)^n + \left(\dfrac{1-\sqrt{5}}{2} \right)^n$,

that is,

(14) $L_n = a^n + b^n$.

The equalities (10) and (14) are called *Binet's Formulas*.

We will use, further, the following notation: if a, b∈**R** and n∈**N**, then:

$$[a,b]_n=\begin{cases} a, \text{if } n \text{ is odd} \\ b, \text{if } n \text{ is even} \end{cases}. \qquad (*)$$

3. The main results

With the above notations, the following statements hold:

Teorema A: *If a is a root of the equation (5), k, m, n∈**N**, the following equalities hold*:

1) $sh(kx)=\frac{1}{2}\cdot[L_{kn}, \sqrt{5}\cdot F_{kn}]_{kn}$, (A.1)

2) $ch(kx)=\frac{1}{2}\cdot[\sqrt{5}\cdot F_{kn}, L_{kn}]_{kn}$, (A.2)

3) $sh(x+y)=\frac{1}{2}\cdot[L_{n+m}, \sqrt{5}\cdot F_{n+m}]_{n+m}$, (A.3)

4) $ch(x+y)=\frac{1}{2}\cdot[\sqrt{5}\cdot F_{n+m}, L_{n+m}]_{n+m}$, (A.4)

5) $sh(x-y)=\frac{1}{2}\cdot[L_{n-m}, \sqrt{5}\cdot F_{n-m}]_{n-m}$, (A.5)

6) $ch(x-y)=\frac{1}{2}\cdot[\sqrt{5}\cdot F_{n-m}, L_{n-m}]_{n-m}$, (A.6)

where:

$x=n\cdot \ln a$ *and* $y=m\cdot \ln a$.

***Proof*: 1)** According to the statement and equality (2.1),

(15) $sh(kx)=\frac{e^{kx}-e^{-kx}}{2}=\frac{e^{kn\cdot\ln a}-e^{-kn\cdot\ln a}}{2}=\frac{e^{\ln a^{kn}}-e^{-\ln a^{kn}}}{2}$

$=\frac{a^{kn}-a^{-kn}}{2}=\frac{a^{kn}-\left(\frac{1}{a}\right)^{kn}}{2}=\frac{a^{kn}-(-b)^{kn}}{2}$

$$=\begin{cases} \frac{1}{2}\cdot(a^{kn}+b^{kn}), \text{if } kn \text{ is odd} \\ \frac{1}{2}\cdot(a^{kn}-b^{kn}), \text{if } kn \text{ is even} \end{cases}.$$

On the other hand, from the equality (14) it follows that:

(16) $L_{kn}=a^{kn}+b^{kn}$,

and from the equalities (9) and (10), it follows that:

(17) $\sqrt{5}\cdot F_{kn}=a^{kn}-b^{kn}$.

Now, from the equalities (∗), (16) and (17), it follows that:

(18) $$\frac{1}{2}\cdot[L_{kn}, \sqrt{5}\cdot F_{kn}]_{kn}=\begin{cases} \frac{1}{2}\cdot L_{kn} \quad, \text{if } kn \text{ is odd} \\ \frac{1}{2}\cdot\sqrt{5}\cdot F_{kn}, \text{if } kn \text{ is even} \end{cases}$$

$$=\begin{cases}\frac{1}{2}\cdot(a^{kn}+b^{kn}), \text{if kn is odd}\\ \frac{1}{2}\cdot(a^{kn}-b^{kn}), \text{if kn is even}\end{cases}.$$

Finally, from the equalities (15) and (18) it follows the equality (A.1).

2) According to the statement and the equality (2.2),

$$\textbf{(19)}\mathrm{ch}(kx)=\frac{e^{kx}+e^{-kx}}{2}=\frac{e^{kn\cdot\ln a}+e^{-kn\cdot\ln a}}{2}=\frac{e^{\ln a^{kn}}+e^{-\ln a^{kn}}}{2}$$

$$=\frac{a^{kn}+a^{-kn}}{2}=\frac{a^{kn}+\left(\frac{1}{a}\right)^{kn}}{2}=\frac{a^{kn}+(-b)^{kn}}{2}$$

$$=\begin{cases}\frac{1}{2}\cdot(a^{kn}-b^{kn}), \text{if kn is odd}\\ \frac{1}{2}\cdot(a^{kn}+b^{kn}), \text{if kn is even}\end{cases}.$$

Now, from the equalities (∗), (16) and (17), it follows that:

$$\textbf{(20)}\frac{1}{2}\cdot[\sqrt{5}\cdot F_{kn},L_{kn}]_{kn}=\begin{cases}\frac{1}{2}\cdot\sqrt{5}\cdot F_{kn}, \text{if kn is odd}\\ \frac{1}{2}\cdot L_{kn} \quad\quad, \text{if kn is even}\end{cases}$$

$$=\begin{cases}\frac{1}{2}\cdot(a^{kn}-b^{kn}), \text{if kn is odd}\\ \frac{1}{2}\cdot(a^{kn}+b^{kn}), \text{if kn is even}\end{cases}.$$

From the equalities (19) and (20) it follows that the equality (A.2) holds.

3) According to the statement and equality (2.1),

$$\textbf{(21)}\mathrm{sh}(x+y)=\frac{e^{x+y}-e^{-x-y}}{2}=\frac{e^{n\cdot\ln a+m\cdot\ln a}-e^{-n\cdot\ln a-m\cdot\ln a}}{2}=\frac{e^{\ln a^{n+m}}-e^{-\ln a^{n+m}}}{2}$$

$$=\frac{a^{n+m}-a^{-n-m}}{2}=\frac{a^{n+m}-\left(\frac{1}{a}\right)^{n+m}}{2}=\frac{a^{n+m}-(-b)^{n+m}}{2}$$

$$=\begin{cases}\frac{1}{2}\cdot(a^{n+m}+b^{n+m}), \text{if } n+m \text{ is odd}\\ \frac{1}{2}\cdot(a^{n+m}-b^{n+m}), \text{if } n+m \text{ is even}\end{cases}.$$

Now, from the equalities (∗), (16) and (17), it follows that:

$$\textbf{(22)}\frac{1}{2}\cdot[L_{n+m},\sqrt{5}\cdot F_{n+m}]_{n+m}=\begin{cases}\frac{1}{2}\cdot L_{n+m} \quad\quad, \text{if } n+m \text{ is odd}\\ \frac{1}{2}\cdot\sqrt{5}\cdot F_{n+m}, \text{if } n+m \text{ is even}\end{cases}$$

$$=\begin{cases}\frac{1}{2}\cdot(a^{n+m}+b^{n+m}), \text{if } n+m \text{ is odd}\\ \frac{1}{2}\cdot(a^{n+m}-b^{n+m}), \text{if } n+m \text{ is even}\end{cases}.$$

From the equalities (21) and (22) it follows that the equality (A.3) holds.
***Otherwise*:** According to the equalities (3.2), (A.1), (A.2) (- for k = 1) and (22),

(23) sh(x+y)=shx·chy+chx·shy

$$=\frac{1}{2}\cdot[L_n, \sqrt{5}\cdot F_n]_n\cdot\frac{1}{2}\cdot[\sqrt{5}\cdot F_m, L_m]_m+\frac{1}{2}\cdot[L_m, \sqrt{5}\cdot F_m]_m\cdot\frac{1}{2}\cdot[\sqrt{5}\cdot F_n, L_n]_n$$

$$=\begin{cases}\frac{1}{4}\cdot(L_n\cdot\sqrt{5}\cdot F_m+L_m\cdot\sqrt{5}\cdot F_n), \text{if n is odd and m is odd}\\ \frac{1}{4}\cdot(L_n\cdot L_m+\sqrt{5}\cdot F_m\cdot\sqrt{5}\cdot F_n), \text{if n is odd and m is even}\\ \frac{1}{4}\cdot(\sqrt{5}\cdot F_n\cdot\sqrt{5}\cdot F_m+L_m\cdot L_n), \text{if n is even and m is odd}\\ \frac{1}{4}\cdot(L_m\cdot\sqrt{5}\cdot F_n+L_n\cdot\sqrt{5}\cdot F_m), \text{if n is even and m is even}\end{cases}$$

$$=\begin{cases}\frac{1}{4}\cdot[(a^n+b^n)\cdot(a^m-b^m)+(a^m+b^m)\cdot(a^n-b^n)], \text{ if n is odd and m is odd}\\ \frac{1}{4}\cdot[(a^n+b^n)\cdot(a^m+b^m)+(a^n-b^n)\cdot(a^m-b^m)], \text{if n is odd and m is even}\\ \frac{1}{4}\cdot[(a^n-b^n)\cdot(a^m-b^m)+(a^m+b^m)\cdot(a^n+b^n)], \text{if n is even and m is odd}\\ \frac{1}{4}\cdot[(a^m+b^m)\cdot(a^n-b^n)+(a^n+b^n)\cdot(a^m-b^m)], \text{if n is even and m is even}\end{cases}$$

$$=\begin{cases}\frac{1}{2}\cdot(a^{n+m}-b^{n+m}), \text{if n is odd and m is odd}\\ \frac{1}{2}\cdot(a^{n+m}+b^{n+m}), \text{if n is odd and m is even}\\ \frac{1}{2}\cdot(a^{n+m}+b^{n+m}), \text{if n is even and m is odd}\\ \frac{1}{2}\cdot(a^{n+m}-b^{n+m}), \text{if n is een and m is even}\end{cases}$$

$$=\begin{cases}\frac{1}{2}\cdot(a^{n+m}-b^{n+m}), \text{if } n+m \text{ is odd}\\ \frac{1}{2}\cdot(a^{n+m}+b^{n+m}), \text{if } n+m \text{ is even}\end{cases}$$

$$=\frac{1}{2}\cdot[L_{n+m}, \sqrt{5}\cdot F_{n+m}]_{n+m}.$$

4) According to the statement and equality (2.2),

(24) $$\text{ch}(x+y)=\frac{e^{x+y}+e^{-x-y}}{2}=\frac{e^{n\cdot\ln a+m\cdot\ln a}+e^{-n\cdot\ln a-m\cdot\ln a}}{2}=\frac{e^{\ln a^{n+m}}+e^{-\ln a^{n+m}}}{2}$$

$$=\frac{a^{n+m}+a^{-n-m}}{2}=\frac{a^{n+m}+\left(\frac{1}{a}\right)^{n+m}}{2}=\frac{a^{n+m}+(-b)^{n+m}}{2}$$

$$=\begin{cases}\frac{1}{2}\cdot(a^{n+m}-b^{n+m}), \text{if } n+m \text{ is odd}\\ \frac{1}{2}\cdot(a^{n+m}+b^{n+m}), \text{if } n+m \text{ is even}\end{cases}.$$

Now, from the equalities (∗), (16) and (17), it follows that:

$$\textbf{(25)}\ \frac{1}{2}\cdot[\sqrt{5}\cdot F_{n+m},L_{n+m}]_{n+m}=\begin{cases}\frac{1}{2}\cdot\sqrt{5}\cdot F_{n+m}, \text{if } n+m \text{ is odd}\\ \frac{1}{2}\cdot L_{n+m} \quad\quad, \text{if } n+m \text{ is even}\end{cases}$$

$$=\begin{cases}\frac{1}{2}\cdot(a^{n+m}-b^{n+m}), \text{if } n+m \text{ is odd}\\ \frac{1}{2}\cdot(a^{n+m}+b^{n+m}), \text{if } n+m \text{ is even}\end{cases}.$$

From the equalities (24) and (25) it follows that the equality (A.4) holds.

***Otherwise*:** According to the equalities (3.4), (A.1), (A.2) (– for k=1) and (25),

(26)ch(x+y)=chx·chy+shx·shy

$$=\frac{1}{2}\cdot[\sqrt{5}\cdot F_n,L_n]_n\cdot\frac{1}{2}\cdot[\sqrt{5}\cdot F_m,L_m]_m+\frac{1}{2}\cdot[L_n,\sqrt{5}\cdot F_n]_n\cdot\frac{1}{2}\cdot[L_m,\sqrt{5}\cdot F_m]_m$$

$$=\begin{cases}\frac{1}{4}\cdot(\sqrt{5}\cdot F_n\cdot\sqrt{5}\cdot F_m+L_n\cdot L_m), \text{if } n \text{ is odd and } m \text{ is odd}\\ \frac{1}{4}\cdot(\sqrt{5}\cdot F_n\cdot L_m+L_n\cdot\sqrt{5}\cdot F_m), \text{if } n \text{ is odd and } m \text{ is even}\\ \frac{1}{4}\cdot(L_n\cdot\sqrt{5}\cdot F_m+\sqrt{5}\cdot F_n\cdot L_m), \text{if } n \text{ is even and } m \text{ is odd}\\ \frac{1}{4}\cdot(L_n\cdot L_m+\sqrt{5}\cdot F_n\cdot\sqrt{5}\cdot F_m), \text{if } n \text{ is even and } m \text{ is even}\end{cases}$$

$$=\begin{cases}\frac{1}{4}\cdot[(a^n-b^n)\cdot(a^m-b^m)+(a^m+b^m)\cdot(a^n-b^n)], \text{if } n \text{ is odd and } m \text{ is odd}\\ \frac{1}{4}\cdot[(a^n-b^n)\cdot(a^m+b^m)+(a^n+b^n)\cdot(a^m-b^m)], \text{if } n \text{ is odd and } m \text{ is even}\\ \frac{1}{4}\cdot[(a^n+b^n)\cdot(a^m-b^m)+(a^m+b^m)\cdot(a^n-b^n)], \text{if } n \text{ is even and } m \text{ is odd}\\ \frac{1}{4}\cdot[(a^m+b^m)\cdot(a^n+b^n)+(a^n-b^n)\cdot(a^m-b^m)], \text{if } n \text{ is even and } m \text{ is even}\end{cases}$$

$$=\begin{cases}\frac{1}{2}\cdot(a^{n+m}+b^{n+m}), \text{if } n \text{ is odd and } m \text{ is odd}\\ \frac{1}{2}\cdot(a^{n+m}-b^{n+m}), \text{if } n \text{ is odd and } m \text{ is even}\\ \frac{1}{2}\cdot(a^{n+m}-b^{n+m}), \text{if } n \text{ is even and } m \text{ is odd}\\ \frac{1}{2}\cdot(a^{n+m}+b^{n+m}), \text{if } n \text{ is even and } m \text{ is even}\end{cases}$$

$$=\begin{cases}\frac{1}{2}\cdot(a^{n+m}+b^{n+m}), \text{if } n+m \text{ is odd}\\ \frac{1}{2}\cdot(a^{n+m}-b^{n+m}), \text{if } n+m \text{ is even}\end{cases}$$

$$=\frac{1}{2}\cdot[\sqrt{5}\cdot F_{n+m}, L_{n+m}]_{n+m}.$$

5) According to the statement and equality (2.1),

(27) $sh(x\text{-}y)=\frac{e^{x-y}-e^{-x+y}}{2}=\frac{e^{n\cdot\ln a-m\cdot\ln a}-e^{-n\cdot\ln a+m\cdot\ln a}}{2}=\frac{e^{\ln a^{n-m}}-e^{-\ln a^{-n+m}}}{2}$

$$=\frac{a^{n-m}-a^{-n+m}}{2}=\frac{a^{n-m}-\left(\frac{1}{a}\right)^{n-m}}{2}=\frac{a^{n-m}-(-b)^{n-m}}{2}$$

$$=\begin{cases}\frac{1}{2}\cdot(a^{n-m}+b^{n-m}), \text{if } n-m \text{ is odd}\\ \frac{1}{2}\cdot(a^{n-m}-b^{n-m}), \text{if } n-m \text{ is even}\end{cases}.$$

Now, from the equalities (∗), (16) and (17), it follows that:

(28) $\frac{1}{2}\cdot[L_{n-m}, \sqrt{5}\cdot F_{n-m}]_{n-m}=\begin{cases}\frac{1}{2}\cdot L_{n-m} &, \text{if } n-m \text{ is odd}\\ \frac{1}{2}\cdot\sqrt{5}\cdot F_{n-m}, & \text{if } n-m \text{ is even}\end{cases}$

$$=\begin{cases}\frac{1}{2}\cdot(a^{n-m}+b^{n-m}), \text{if } n-m \text{ is odd}\\ \frac{1}{2}\cdot(a^{n-m}-b^{n-m}), \text{if } n-m \text{ is even}\end{cases}.$$

From the equalities (27) and (28) it follows that the equality (A.5) holds.

***Otherwise*:** According to the equalities (3.4), (A.1), (A.2) (– for k=1) and (28),

(29) $sh(x\text{-}y)=shx\cdot chy\text{-}chx\cdot shy$

$$=\frac{1}{2}\cdot[L_n, \sqrt{5}\cdot F_n]_n\cdot\frac{1}{2}\cdot[\sqrt{5}\cdot F_m, L_m]_m-\frac{1}{2}\cdot[L_m, \sqrt{5}\cdot F_m]_m\cdot\frac{1}{2}\cdot[\sqrt{5}\cdot F_n, L_n]_n$$

$$=\begin{cases}\frac{1}{4}\cdot(L_n\cdot\sqrt{5}\cdot F_m-L_m\cdot\sqrt{5}\cdot F_n), \text{if } n \text{ is odd and } m \text{ is odd}\\ \frac{1}{4}\cdot(L_n\cdot L_m-\sqrt{5}\cdot F_m\cdot\sqrt{5}\cdot F_n), \text{if } n \text{ is odd and } m \text{ is even}\\ \frac{1}{4}\cdot(\sqrt{5}\cdot F_n\cdot\sqrt{5}\cdot F_m-L_m\cdot L_n), \text{if } n \text{ is even and } m \text{ is odd}\\ \frac{1}{4}\cdot(L_m\cdot\sqrt{5}\cdot F_n-L_n\cdot\sqrt{5}\cdot F_m), \text{if } n \text{ is even and } m \text{ is even}\end{cases}$$

$$=\begin{cases}\frac{1}{4}\cdot[(a^n+b^n)\cdot(a^m-b^m)-(a^m+b^m)\cdot(a^n-b^n)], \text{if } n \text{ is odd and } m \text{ is odd}\\ \frac{1}{4}\cdot[(a^n+b^n)\cdot(a^m+b^m)-(a^n-b^n)\cdot(a^m-b^m)], \text{if } n \text{ is odd and } m \text{ is even}\\ \frac{1}{4}\cdot[(a^n-b^n)\cdot(a^m-b^m)-(a^m+b^m)\cdot(a^n+b^n)], \text{if } n \text{ is even and } m \text{ is odd}\\ \frac{1}{4}\cdot[(a^m+b^m)\cdot(a^n-b^n)-(a^n+b^n)\cdot(a^m-b^m)], \text{if } n \text{ is even and } m \text{ is even}\end{cases}$$

$$=\begin{cases}\frac{1}{2}\cdot(a^m\cdot b^n-a^n\cdot b^m)\ \ ,\text{if n is odd and m is odd}\\ \frac{1}{2}\cdot(a^m\cdot b^n+a^n\cdot b^m)\ \ ,\text{if n is odd and m is even}\\ \frac{1}{2}\cdot(-a^n\cdot b^m-a^m\cdot b^n),\text{if n is even and is odd}\\ \frac{1}{2}\cdot(-a^m\cdot b^n+a^n\cdot b^m),\text{if n is even and m is even}\end{cases}$$

$$=\begin{cases}\frac{1}{2}\cdot\left(\frac{-a^m\cdot b^n+a^n\cdot b^m}{(-1)^m}\right),\text{if n is odd and m is odd}\\ \frac{1}{2}\cdot\left(\frac{a^m\cdot b^n+a^n\cdot b^m}{(-1)^m}\right)\ ,\text{if n is odd and m is even}\\ \frac{1}{2}\cdot\left(\frac{a^m\cdot b^n+a^n\cdot b^m}{(-1)^m}\right)\ ,\text{if n is even and m is odd}\\ \frac{1}{2}\cdot\left(\frac{-a^m\cdot b^n+a^n\cdot b^m}{(-1)^m}\right),\text{if n is even and m is even}\end{cases}$$

$$=\begin{cases}\frac{1}{2}\cdot\left(\frac{-a^m\cdot b^n+a^n\cdot b^m}{a^m\cdot b^m}\right),\text{if n is odd and m is odd}\\ \frac{1}{2}\cdot\left(\frac{a^m\cdot b^n+a^n\cdot b^m}{a^m\cdot b^m}\right)\ ,\text{if n is odd amd m is even}\\ \frac{1}{2}\cdot\left(\frac{a^m\cdot b^n+a^n\cdot b^m}{a^m\cdot b^m}\right)\ ,\text{if n is even and m is odd}\\ \frac{1}{2}\cdot\left(\frac{-a^m\cdot b^n+a^n\cdot b^m}{a^m\cdot b^m}\right),\text{if n is even and m is even}\end{cases}$$

$$=\begin{cases}\frac{1}{2}\cdot(a^{n-m}-b^{n-m}),\text{if n is odd and m is odd}\\ \frac{1}{2}\cdot(a^{n-m}+b^{n-m}),\text{if n is odd and m is even}\\ \frac{1}{2}\cdot(a^{n-m}+b^{n-m}),\text{if n is even and m is odd}\\ \frac{1}{2}\cdot(a^{n-m}-b^{n-m}),\text{if n is even and m is even}\end{cases}$$

$$=\begin{cases}\frac{1}{2}\cdot(a^{n-m}+b^{n-m}),\text{if n - m is odd}\\ \frac{1}{2}\cdot(a^{n-m}-b^{n-m}),\text{if n - m is even}\end{cases}$$

$$=\frac{1}{2}\cdot[L_{n-m},\sqrt{5}\cdot F_{n-m}]_{n-m}.$$

6) According to the statement and equality (2.2),

$$\textbf{(30)}\mathrm{ch}(x\text{-}y)=\frac{e^{x-y}+e^{-x+y}}{2}=\frac{e^{n\cdot\ln a-m\cdot\ln a}+e^{-n\cdot\ln a+m\cdot\ln a}}{2}=\frac{e^{\ln a^{n-m}}+e^{-\ln a^{n-m}}}{2}$$

$$=\frac{a^{n-m}+a^{-n+m}}{2}=\frac{a^{n-m}+\left(\frac{1}{a}\right)^{n-m}}{2}=\frac{a^{n-m}+(-b)^{n-m}}{2}$$

$$=\begin{cases}\frac{1}{2}\cdot(a^{n-m}-b^{n-m}), \text{if } n-m \text{ is odd}\\ \frac{1}{2}\cdot(a^{n-m}+b^{n-m}), \text{if } n-m \text{ is even}\end{cases}.$$

Now, from the equalities (∗), (16) and (17), it follows that:

$$\textbf{(31)}\ \frac{1}{2}\cdot[\sqrt{5}\cdot F_{n-m},L_{n-m}]_{n-m}=\begin{cases}\frac{1}{2}\cdot\sqrt{5}\cdot F_{n-m}, \text{if } n-m \text{ is odd}\\ \frac{1}{2}\cdot L_{n-m} \quad, \text{if } n-m \text{ is even}\end{cases}$$

$$=\begin{cases}\frac{1}{2}\cdot(a^{n-m}-b^{n-m}), \text{if } n-m \text{ is odd}\\ \frac{1}{2}\cdot(a^{n-m}+b^{n-m}), \text{if } n-m \text{ is even}\end{cases}.$$

From the equalities (30) and (31) it follows that the equality (A.6) holds.

***Otherwise*:** According to the equalities (3.5), (A.1), (A.2) (– for k=1) and (31),

(32)ch(x-y)=chx·chy-shx·shy

$$=\frac{1}{2}\cdot[\sqrt{5}\cdot F_n,L_n]_n\cdot\frac{1}{2}\cdot[\sqrt{5}\cdot F_m,L_m]_m-\frac{1}{2}\cdot[L_n,\sqrt{5}\cdot F_n]_n\cdot\frac{1}{2}\cdot[L_m,\sqrt{5}\cdot F_m]_m$$

$$=\begin{cases}\frac{1}{4}\cdot(\sqrt{5}\cdot F_n\cdot\sqrt{5}\cdot F_m-L_n\cdot L_m), \text{if } n \text{ is odd and } m \text{ is odd}\\ \frac{1}{4}\cdot(\sqrt{5}\cdot F_n\cdot L_m-L_n\cdot\sqrt{5}\cdot F_m), \text{if } n \text{ is odd and } m \text{ is even}\\ \frac{1}{4}\cdot(L_n\cdot\sqrt{5}\cdot F_m-\sqrt{5}\cdot F_n\cdot L_m), \text{if } n \text{ is even and } m \text{ is odd}\\ \frac{1}{4}\cdot(L_n\cdot L_m-\sqrt{5}\cdot F_n\cdot\sqrt{5}\cdot F_m), \text{if } n \text{ is even and } m \text{ is even}\end{cases}$$

$$=\begin{cases}\frac{1}{4}\cdot[(a^n-b^n)\cdot(a^m-b^m)-(a^m+b^m)\cdot(a^n-b^n)], \text{if } n \text{ is odd and } m \text{ is odd}\\ \frac{1}{4}\cdot[(a^n-b^n)\cdot(a^m+b^m)-(a^n+b^n)\cdot(a^m-b^m)], \text{if } n \text{ is odd and } m \text{ is even}\\ \frac{1}{4}\cdot[(a^n+b^n)\cdot(a^m-b^m)-(a^m+b^m)\cdot(a^n-b^n)], \text{if } n \text{ is even and } m \text{ is odd}\\ \frac{1}{4}\cdot[(a^m+b^m)\cdot(a^n+b^n)-(a^n-b^n)\cdot(a^m-b^m)], \text{if } n \text{ is even and } m \text{ is even}\end{cases}$$

$$=\begin{cases}\frac{1}{2}\cdot(-a^n\cdot b^m-a^m\cdot b^n), \text{if } n \text{ is odd and } m \text{ is odd}\\ \frac{1}{2}\cdot(a^n\cdot b^m-a^m\cdot b^n)\ , \text{if } n \text{ is odd and } m \text{ is even}\\ \frac{1}{2}\cdot(-a^n\cdot b^m+a^m\cdot b^n), \text{if } n \text{ is even and } m \text{ is odd}\\ \frac{1}{2}\cdot(a^n\cdot b^m+a^m\cdot b^n)\ , \text{if } n \text{ is even and } m \text{ is even}\end{cases}$$

$$=\begin{cases}\frac{1}{2}\cdot\left(\frac{a^m\cdot b^n+a^n\cdot b^m}{(-1)^m}\right) & \text{, if n is odd and m is odd}\\ \frac{1}{2}\cdot\left(\frac{-a^m\cdot b^n+a^n\cdot b^m}{(-1)^m}\right) & \text{, if n is odd and m is even}\\ \frac{1}{2}\cdot\left(\frac{-a^m\cdot b^n+a^n\cdot b^m}{(-1)^m}\right) & \text{, if n is even and m is odd}\\ \frac{1}{2}\cdot\left(\frac{a^m\cdot b^n+a^n\cdot b^m}{(-1)^m}\right) & \text{, if n is even and m is even}\end{cases}$$

$$=\begin{cases}\frac{1}{2}\cdot\left(\frac{a^m\cdot b^n+a^n\cdot b^m}{a^m\cdot b^m}\right) & \text{, if n is odd and m is odd}\\ \frac{1}{2}\cdot\left(\frac{-a^m\cdot b^n+a^n\cdot b^m}{a^m\cdot b^m}\right) & \text{, if n is odd and m is even}\\ \frac{1}{2}\cdot\left(\frac{-a^m\cdot b^n+a^n\cdot b^m}{a^m\cdot b^m}\right) & \text{, if n is even and m is odd}\\ \frac{1}{2}\cdot\left(\frac{a^m\cdot b^n+a^n\cdot b^m}{a^m\cdot b^m}\right) & \text{, if n is even and m is even}\end{cases}$$

$$=\begin{cases}\frac{1}{2}\cdot(a^{n-m}+b^{n-m}), \text{if n is odd and m is odd}\\ \frac{1}{2}\cdot(a^{n-m}-b^{n-m}), \text{if n is odd and m is even}\\ \frac{1}{2}\cdot(a^{n-m}-b^{n-m}), \text{if n is even and m is odd}\\ \frac{1}{2}\cdot(a^{n-m}+b^{n-m}), \text{if n is even and m is even}\end{cases}$$

$$=\begin{cases}\frac{1}{2}\cdot(a^{n-m}-b^{n-m}), \text{if n - m is odd}\\ \frac{1}{2}\cdot(a^{n-m}+b^{n-m}), \text{if n - m is even}\end{cases}$$

$$=\frac{1}{2}\cdot[\sqrt{5}\cdot F_{n-m},L_{n-m}]_{n-m}.\ \square$$

The following results are consequences of the above.

Corollary B: *With the notations in Theorem A, the following equalities hold*:

7) $th(kx)=\left[\frac{L_{kn}}{\sqrt{5}\cdot F_{kn}},\frac{\sqrt{5}\cdot F_{kn}}{L_{kn}}\right]_{kn}$; (B.1)

8) $cth(kx)=\left[\frac{\sqrt{5}\cdot F_{kn}}{L_{kn}},\frac{L_{kn}}{\sqrt{5}\cdot F_{kn}}\right]_{kn}$; (B.2)

9) $sch(kx)=\left[\frac{2}{\sqrt{5}\cdot F_{kn}},\frac{2}{L_{kn}}\right]_{kn}$; (B.3)

10) $csh(kx)=\left[\frac{2}{L_{kn}},\frac{2}{\sqrt{5}\cdot F_{kn}}\right]_{kn}$; (B.4)

11) $th(x+y)=\left[\dfrac{L_{n+m}}{\sqrt{5}\cdot F_{n+m}},\dfrac{\sqrt{5}\cdot F_{n+m}}{L_{n+m}}\right]_{n+m}$; (B.5)

12) $cth(x+y)=\left[\dfrac{\sqrt{5}\cdot F_{n+m}}{L_{n+m}},\dfrac{L_{n+m}}{\sqrt{5}\cdot F_{n+m}}\right]_{n+m}$; (B.6)

13) $sch(x+y)=\left[\dfrac{2}{\sqrt{5}\cdot F_{n+m}},\dfrac{2}{L_{n+m}}\right]_{n+m}$; (B.7)

14) $csh(x+y)=\left[\dfrac{2}{L_{n+m}},\dfrac{2}{\sqrt{5}\cdot F_{n+m}}\right]_{n+m}$; (B.8)

15) $th(x-y)=\left[\dfrac{L_{n-m}}{\sqrt{5}\cdot F_{n-m}},\dfrac{\sqrt{5}\cdot F_{n-m}}{L_{n-m}}\right]_{n-m}$; (B.9)

16) $cth(x-y)=\left[\dfrac{\sqrt{5}\cdot F_{n-m}}{L_{n-m}},\dfrac{L_{n-m}}{\sqrt{5}\cdot F_{n-m}}\right]_{n-m}$; (B.10)

17) $sch(x-y)=\left[\dfrac{2}{\sqrt{5}\cdot F_{n-m}},\dfrac{2}{L_{n-m}}\right]_{n-m}$; (B.11)

18) $csh(x-y)=\left[\dfrac{2}{L_{n-m}},\dfrac{2}{\sqrt{5}\cdot F_{n-m}}\right]_{n-m}$. (B.12)

***Proof*: 7)** According to hypothesis and equalities (2.3), (A.1), (A.2) and (∗),

$$\textbf{(33)}\mathrm{th}(kx)=\frac{\mathrm{sh}(kx)}{\mathrm{ch}(kx)}=\frac{\left[L_{kn},\sqrt{5}\cdot F_{kn}\right]_{kn}}{\left[\sqrt{5}\cdot F_{kn},L_{kn}\right]_{kn}}$$

$$=\begin{cases}\dfrac{L_{kn}}{\sqrt{5}\cdot F_{kn}}, \text{if kn is odd}\\ \dfrac{\sqrt{5}\cdot F_{kn}}{L_{kn}}, \text{if kn is even}\end{cases}$$

$$=\left[\frac{L_{kn}}{\sqrt{5}\cdot F_{kn}},\frac{\sqrt{5}\cdot F_{kn}}{L_{kn}}\right]_{kn};$$

which shows that the equality (B.1) holds.

***Otherwise*:** According to hypothesis and equalities (2.3′), (16) and (17),

$$\textbf{(34)}\mathrm{th}(kx)=\frac{e^{kx}-e^{-kx}}{e^{kx}+e^{-kx}}=\frac{e^{kn\cdot\ln a}-e^{-kn\cdot\ln a}}{e^{kn\cdot\ln a}+e^{-kn\cdot\ln a}}=\frac{e^{\ln a^{kn}}-e^{-\ln a^{kn}}}{e^{\ln a^{kn}}+e^{-\ln a^{kn}}}$$

$$=\frac{a^{kn}-a^{-kn}}{a^{kn}+a^{-kn}}=\frac{a^{kn}-\left(\frac{1}{a}\right)^{kn}}{a^{kn}+\left(\frac{1}{a}\right)^{kn}}=\frac{a^{kn}-(-b)^{kn}}{a^{kn}+(-b)^{kn}}$$

$$=\begin{cases}\dfrac{a^{kn}+b^{kn}}{a^{kn}-b^{kn}}, \text{if kn is odd}\\ \dfrac{a^{kn}-b^{kn}}{a^{kn}+b^{kn}}, \text{if kn is even}\end{cases}$$

$$=\begin{cases}\dfrac{L_{kn}}{\sqrt{5}\cdot F_{kn}}, \text{if kn is odd}\\ \dfrac{\sqrt{5}\cdot F_{kn}}{L_{kn}}, \text{if kn is even}\end{cases}$$

$$=\left[\frac{L_{kn}}{\sqrt{5}\cdot F_{kn}},\frac{\sqrt{5}\cdot F_{kn}}{L_{kn}}\right]_{kn};$$

therefore, we obtain again the equality (B.1).

8) According to hypothesis and equalities (2.4), (A.1), (A.2) and (∗),

$$(\mathbf{35})\operatorname{cth}(kx)=\frac{\operatorname{ch}(kx)}{\operatorname{sh}(kx)}=\frac{\left[\sqrt{5}\cdot F_{kn},L_{kn}\right]_{kn}}{\left[L_{kn},\sqrt{5}\cdot F_{kn}\right]_{kn}}$$

$$=\begin{cases}\dfrac{\sqrt{5}\cdot F_{kn}}{L_{kn}}, \text{if kn is odd}\\ \dfrac{L_{kn}}{\sqrt{5}\cdot F_{kn}}, \text{if kn is even}\end{cases}$$

$$=\left[\frac{\sqrt{5}\cdot F_{kn}}{L_{kn}},\frac{L_{kn}}{\sqrt{5}\cdot F_{kn}}\right]_{kn};$$

which shows that the equality (B.2) holds.

***Otherwise*:** According to hypothesis and equalities (2.4′), (16) and (17),

$$(\mathbf{36})\operatorname{cth}(kx)=\frac{e^{kx}+e^{-kx}}{e^{kx}-e^{-kx}}=\frac{e^{kn\cdot\ln a}+e^{-kn\cdot\ln a}}{e^{kn\cdot\ln a}-e^{-kn\cdot\ln a}}=\frac{e^{\ln a^{kn}}+e^{-\ln a^{kn}}}{e^{\ln a^{kn}}-e^{-\ln a^{kn}}}$$

$$=\frac{a^{kn}+a^{-kn}}{a^{kn}-a^{-kn}}=\frac{a^{kn}+\left(\frac{1}{a}\right)^{kn}}{a^{kn}-\left(\frac{1}{a}\right)^{kn}}=\frac{a^{kn}+(-b)^{kn}}{a^{kn}-(-b)^{kn}}$$

$$=\begin{cases}\dfrac{a^{kn}-b^{kn}}{a^{kn}+b^{kn}}, \text{if kn is odd}\\ \dfrac{a^{kn}+b^{kn}}{a^{kn}-b^{kn}}, \text{if kn is even}\end{cases}$$

$$=\begin{cases}\dfrac{\sqrt{5}\cdot F_{kn}}{L_{kn}}, \text{if kn is odd}\\ \dfrac{L_{kn}}{\sqrt{5}\cdot F_{kn}}, \text{if kn is even}\end{cases}$$

$$=\left[\frac{\sqrt{5}\cdot F_{kn}}{L_{kn}},\frac{L_{kn}}{\sqrt{5}\cdot F_{kn}}\right]_{kn};$$

therefore, we obtain again the equality (B.2).

9) According to hypothesis and equalities (2.5), (A.2) and (∗),

$$(\mathbf{37})\operatorname{sch}(kx)=\frac{1}{\operatorname{ch}(kx)}=\frac{2}{\left[\sqrt{5}\cdot F_{kn},L_{kn}\right]_{kn}}$$

$$=\begin{cases}\dfrac{2}{\sqrt{5}\cdot F_{kn}}, \text{if kn is odd}\\ \dfrac{2}{L_{kn}}\quad, \text{if kn is even}\end{cases}$$

$$=\left[\frac{2}{\sqrt{5}\cdot F_{kn}}, \frac{2}{L_{kn}}\right]_{kn};$$

which shows that the equality (B.3) holds.

Otherwise**:** According to hypothesis and equalities (2.5′), (16) and (17),

$$\textbf{(38)}\text{sch}(kx)=\frac{2}{e^{kx}+e^{-kx}}=\frac{2}{e^{kn\cdot\ln a}+e^{-kn\cdot\ln a}}=\frac{2}{e^{\ln a^{kn}}+e^{-\ln a^{kn}}}$$

$$=\frac{2}{a^{kn}+a^{-kn}}=\frac{2}{a^{kn}+\left(\frac{1}{a}\right)^{kn}}=\frac{2}{a^{kn}+(-b)^{kn}}$$

$$=\begin{cases}\dfrac{2}{a^{kn}-b^{kn}}, \text{if kn is odd}\\ \dfrac{2}{a^{kn}+b^{kn}}, \text{if kn is even}\end{cases}$$

$$=\begin{cases}\dfrac{2}{\sqrt{5}\cdot F_{kn}}, \text{if kn is odd}\\ \dfrac{2}{L_{kn}}\quad, \text{if kn is even}\end{cases}$$

$$=\left[\frac{2}{\sqrt{5}\cdot F_{kn}}, \frac{2}{L_{kn}}\right]_{kn};$$

therefore, we obtain again the equality (B.3).

10) According to hypothesis and equalities (2.6), (A.1) and (∗),

$$\textbf{(39)}\text{csh}(kx)=\frac{1}{\text{sh}(kx)}=\frac{2}{\left[L_{kn}, \sqrt{5}\cdot F_{kn}\right]_{kn}}$$

$$=\begin{cases}\dfrac{2}{L_{kn}}\quad, \text{if kn is odd}\\ \dfrac{2}{\sqrt{5}\cdot F_{kn}}, \text{if kn is even}\end{cases}$$

$$=\left[\frac{2}{L_{kn}}, \frac{2}{\sqrt{5}\cdot F_{kn}}\right]_{kn};$$

which shows that the equality (B.4) holds.

Otherwise**:** According to hypothesis and equalities (2.6′), (16) and (17),

$$\textbf{(40)}\text{csh}(kx)=\frac{2}{e^{kx}-e^{-kx}}=\frac{2}{e^{kn\cdot\ln a}-e^{-kn\cdot\ln a}}=\frac{2}{e^{\ln a^{kn}}-e^{-\ln a^{kn}}}$$

$$=\frac{2}{a^{kn}-a^{-kn}}=\frac{2}{a^{kn}-\left(\frac{1}{a}\right)^{kn}}=\frac{2}{a^{kn}-(-b)^{kn}}$$

$$=\begin{cases}\dfrac{2}{a^{kn}+b^{kn}}, \text{if kn is odd}\\ \dfrac{2}{a^{kn}-b^{kn}}, \text{if kn is even}\end{cases}$$

$$=\begin{cases}\dfrac{2}{L_{kn}} \quad , \text{if kn is odd}\\ \dfrac{2}{\sqrt{5}\cdot F_{kn}}, \text{if kn is even}\end{cases}$$

$$=\left[\frac{2}{L_{kn}}, \frac{2}{\sqrt{5}\cdot F_{kn}}\right]_{kn} ;$$

therefore, we obtain again the equality (B.4).

11) According to hypothesis and equalities (2.3), (A.3) and (A.4),

$$\textbf{(41)}\text{th}(x+y)=\frac{\text{sh}(x+y)}{\text{ch}(x+y)}=\frac{\left[L_{n+m}, \sqrt{5}\cdot F_{n+m}\right]_{n+m}}{\left[\sqrt{5}\cdot F_{n+m}, L_{n+m}\right]_{n+m}}$$

$$=\begin{cases}\dfrac{L_{n+m}}{\sqrt{5}\cdot F_{n+m}}, \text{if } n+m \text{ is odd}\\ \dfrac{\sqrt{5}\cdot F_{n+m}}{L_{n+m}}, \text{if } n+m \text{ is even}\end{cases}$$

$$=\left[\frac{L_{n+m}}{\sqrt{5}\cdot F_{n+m}}, \frac{\sqrt{5}\cdot F_{n+m}}{L_{n+m}}\right]_{n+m} ;$$

which shows that the equality (B.5) holds.

Otherwise**:** According to hypothesis and equalities (2.3′), (16) and (17),

$$\textbf{(42)}\text{th}(x+y)=\frac{e^{x+y}-e^{-x-y}}{e^{x+y}+e^{-x-y}}=\frac{e^{n\cdot\ln a+m\cdot\ln a}-e^{-n\cdot\ln a-m\cdot\ln a}}{e^{n\cdot\ln a+m\cdot\ln a}+e^{-n\cdot\ln a-m\cdot\ln a}}=\frac{e^{\ln a^{n+m}}-e^{-\ln a^{n+m}}}{e^{\ln a^{n+m}}+e^{-\ln a^{n+m}}}$$

$$=\frac{a^{n+m}-a^{-n-m}}{a^{n+m}+a^{-n-m}}=\frac{a^{n+m}-\left(\frac{1}{a}\right)^{n+m}}{a^{n+m}+\left(\frac{1}{a}\right)^{n+m}}=\frac{a^{n+m}-(-b)^{n+m}}{a^{n+m}+(-b)^{n+m}}$$

$$=\begin{cases}\dfrac{a^{n+m}+b^{n+m}}{a^{n+m}-b^{n+m}}, \text{if } n+m \text{ is odd}\\ \dfrac{a^{n+m}-b^{n+m}}{a^{n+m}+b^{n+m}}, \text{if } n+m \text{ is even}\end{cases}$$

$$=\begin{cases}\dfrac{L_{n+m}}{\sqrt{5}\cdot F_{n+m}}, \text{if } n+m \text{ is odd}\\ \dfrac{\sqrt{5}\cdot F_{n+m}}{L_{n+m}}, \text{if } n+m \text{ is even}\end{cases}$$

$$=\left[\frac{L_{n+m}}{\sqrt{5}\cdot F_{n+m}}, \frac{\sqrt{5}\cdot F_{n+m}}{L_{n+m}}\right]_{n+m} ;$$

therefore, we obtain again the equality (B.5).

Otherwise**:** According to hypothesis and equalities (3.6), (B.1), (16) and (17),

(43) $\mathrm{th}(x+y) = \dfrac{\mathrm{th}x + \mathrm{th}y}{1 + \mathrm{th}x \cdot \mathrm{th}y} = \dfrac{\left[\dfrac{L_n}{\sqrt{5}\cdot F_n}, \dfrac{\sqrt{5}\cdot F_n}{L_n}\right]_n + \left[\dfrac{L_m}{\sqrt{5}\cdot F_m}, \dfrac{\sqrt{5}\cdot F_m}{L_m}\right]_m}{1 + \left[\dfrac{L_n}{\sqrt{5}\cdot F_n}, \dfrac{\sqrt{5}\cdot F_n}{L_n}\right]_n \cdot \left[\dfrac{L_m}{\sqrt{5}\cdot F_m}, \dfrac{\sqrt{5}\cdot F_m}{L_m}\right]_m}$

$$= \begin{cases} \dfrac{\dfrac{L_n}{\sqrt{5}\cdot F_n} + \dfrac{L_m}{\sqrt{5}\cdot F_m}}{1 + \dfrac{L_n}{\sqrt{5}\cdot F_n} \cdot \dfrac{L_m}{\sqrt{5}\cdot F_m}}, \text{if } n \text{ is odd and } m \text{ is odd} \\ \dfrac{\dfrac{L_n}{\sqrt{5}\cdot F_n} + \dfrac{\sqrt{5}\cdot F_m}{L_m}}{1 + \dfrac{L_n}{\sqrt{5}\cdot F_n} \cdot \dfrac{\sqrt{5}\cdot F_m}{L_m}}, \text{if } n \text{ is odd and } m \text{ is even} \\ \dfrac{\dfrac{\sqrt{5}\cdot F_n}{L_n} + \dfrac{L_m}{\sqrt{5}\cdot F_m}}{1 + \dfrac{\sqrt{5}\cdot F_n}{L_n} \cdot \dfrac{L_m}{\sqrt{5}\cdot F_m}}, \text{if } n \text{ is even and } m \text{ is odd} \\ \dfrac{\dfrac{\sqrt{5}\cdot F_n}{L_n} + \dfrac{\sqrt{5}\cdot F_m}{L_m}}{1 + \dfrac{\sqrt{5}\cdot F_n}{L_n} \cdot \dfrac{\sqrt{5}\cdot F_m}{L_m}}, \text{if } n \text{ is even and } m \text{ is even} \end{cases}$$

$$b = \begin{cases} \dfrac{\dfrac{a^n + b^n}{a^n - b^n} + \dfrac{a^m + b^m}{a^m - b^m}}{1 + \dfrac{a^n + b^n}{a^n - b^n} \cdot \dfrac{a^m + b^m}{a^m - b^m}}, \text{if } n \text{ is odd and } m \text{ is odd} \\ \dfrac{\dfrac{a^n + b^n}{a^n - b^n} + \dfrac{a^m - b^m}{a^m + b^m}}{1 + \dfrac{a^n + b^n}{a^n - b^n} \cdot \dfrac{a^m - b^m}{a^m + b^m}}, \text{if } n \text{ is odd and } m \text{ is even} \\ \dfrac{\dfrac{a^n - b^n}{a^n + b^n} + \dfrac{a^m + b^m}{a^m - b^m}}{1 + \dfrac{a^n - b^n}{a^n + b^n} \cdot \dfrac{a^m + b^m}{a^m - b^m}}, \text{if } n \text{ is even and } m \text{ is odd} \\ \dfrac{\dfrac{a^n - b^n}{a^n + b^n} + \dfrac{a^m - b^m}{a^m + b^m}}{1 + \dfrac{a^n - b^n}{a^n + b^n} \cdot \dfrac{a^m - b^m}{a^m + b^m}}, \text{if } n \text{ isi even and } m \text{ is even} \end{cases}$$

$$=\begin{cases}\dfrac{(a^n+b^n)\cdot(a^m-b^m)+(a^n-b^n)\cdot(a^m+b^m)}{(a^n-b^n)\cdot(a^m-b^m)+(a^n+b^n)\cdot(a^m+b^m)}, \text{if n is odd and m is odd}\\ \dfrac{(a^n+b^n)\cdot(a^m+b^m)+(a^n-b^n)\cdot(a^m-b^m)}{(a^n-b^n)\cdot(a^m+b^m)+(a^n+b^n)\cdot(a^m-b^m)}, \text{if n is odd and m is even}\\ \dfrac{(a^n-b^n)\cdot(a^m-b^m)+(a^n+b^n)\cdot(a^m+b^m)}{(a^n+b^n)\cdot(a^m-b^m)+(a^n-b^n)\cdot(a^m+b^m)}, \text{if n is even and m is odd}\\ \dfrac{(a^n-b^n)\cdot(a^m+b^m)+(a^n+b^n)\cdot(a^m-b^m)}{(a^n+b^n)\cdot(a^m+b^m)+(a^n-b^n)\cdot(a^m-b^m)}, \text{if n is even and m is even}\end{cases}$$

$$=\begin{cases}\dfrac{a^{n+m}-b^{n+m}}{a^{n+m}+b^{n+m}}, \text{if n is odd and m is odd}\\ \dfrac{a^{n+m}+b^{n+m}}{a^{n+m}-b^{n+m}}, \text{if n is odd and m is even}\\ \dfrac{a^{n+m}+b^{n+m}}{a^{n+m}-b^{n+m}}, \text{if n is even and m is odd}\\ \dfrac{a^{n+m}-b^{n+m}}{a^{n+m}+b^{n+m}}, \text{if n is even and m is even}\end{cases}$$

$$=\begin{cases}\dfrac{\sqrt{5}\cdot F_{n+m}}{L_{n+m}}, \text{if n is odd and m is odd}\\ \dfrac{L_{n+m}}{\sqrt{5}\cdot F_{n+m}}, \text{if n is odd and m is evenr}\\ \dfrac{L_{n+m}}{\sqrt{5}\cdot F_{n+m}}, \text{if n is even and m is odd}\\ \dfrac{\sqrt{5}\cdot F_{n+m}}{L_{n+m}}, \text{if n is even and m is even}\end{cases}$$

$$=\begin{cases}\dfrac{\sqrt{5}\cdot F_{n+m}}{L_{n+m}}, \text{if } n+m \text{ is even}\\ \dfrac{L_{n+m}}{\sqrt{5}\cdot F_{n+m}}, \text{if } n+m \text{ is odd}\end{cases}$$

$$=\left[\frac{L_{n+m}}{\sqrt{5}\cdot F_{n+m}}, \frac{\sqrt{5}\cdot F_{n+m}}{L_{n+m}}\right]_{n+m};$$

therefore, we obtain again the equality (B.5).

12) According to hypothesis and equalities (2.4), (A.3) and (A.4),

$$\textbf{(44)}\text{cth}(x+y)=\frac{ch(x+y)}{sh(x+y)}=\frac{\left[\sqrt{5}\cdot F_{n+m}, L_{n+m}\right]_{n+m}}{\left[L_{n+m}, \sqrt{5}\cdot F_{n+m}\right]_{n+m}}$$

$$=\begin{cases}\dfrac{\sqrt{5}\cdot F_{n+m}}{L_{n+m}}, \text{if } n+m \text{ is odd}\\ \dfrac{L_{n+m}}{\sqrt{5}\cdot F_{n+m}}, \text{if } n+m \text{ is even}\end{cases}$$

$$=\left[\frac{\sqrt{5}\cdot F_{n+m}}{L_{n+m}}, \frac{L_{n+m}}{\sqrt{5}\cdot F_{n+m}}\right]_{n+m};$$

which shows that the equality (B.6) holds.

***Otherwise*:** According to hypothesis and equalities (2.4′), (16) and (17),

$$(\mathbf{45})\,\mathrm{cth}(x+y)=\frac{e^{x+y}+e^{-x-y}}{e^{x+y}-e^{-x-y}}=\frac{e^{n\cdot\ln a+m\cdot\ln a}+e^{-n\cdot\ln a-m\cdot\ln a}}{e^{n\cdot\ln a+m\cdot\ln a}-e^{-n\cdot\ln a-m\cdot\ln a}}=\frac{e^{\ln a^{n+m}}+e^{-\ln a^{n+m}}}{e^{\ln a^{n+m}}-e^{-\ln a^{n+m}}}$$

$$=\frac{a^{n+m}+a^{-n-m}}{a^{n+m}-a^{-n-m}}=\frac{a^{n+m}+\left(\frac{1}{a}\right)^{n+m}}{a^{n+m}-\left(\frac{1}{a}\right)^{n+m}}=\frac{a^{n+m}+(-b)^{n+m}}{a^{n+m}-(-b)^{n+m}}$$

$$=\begin{cases}\dfrac{a^{n+m}-b^{n+m}}{a^{n+m}+b^{n+m}},\text{ if } n+m \text{ is odd}\\[2ex]\dfrac{a^{n+m}-b^{n+m}}{a^{n+m}+b^{n+m}},\text{ if } n+m \text{ is even}\end{cases}$$

$$=\begin{cases}\dfrac{\sqrt{5}\cdot F_{n+m}}{L_{n+m}},\text{ if } n+m \text{ is odd}\\[2ex]\dfrac{L_{n+m}}{\sqrt{5}\cdot F_{n+m}},\text{ if } n+m \text{ is even}\end{cases}$$

$$=\left[\frac{\sqrt{5}\cdot F_{n+m}}{L_{n+m}},\frac{L_{n+m}}{\sqrt{5}\cdot F_{n+m}}\right]_{n+m};$$

therefore, we obtain again the equality (B.6).

***Otherwise*:** According to hypothesis and equalities (3.8), (B.1), (16) and (17),

$$(\mathbf{46})\,\mathrm{cth}(x+y)=\frac{\mathrm{cth}x\cdot\mathrm{cth}y+1}{\mathrm{cth}x+\mathrm{cth}y}=\frac{\left[\frac{L_n}{\sqrt{5}\cdot F_n},\frac{\sqrt{5}\cdot F_n}{L_n}\right]_n\cdot\left[\frac{L_m}{\sqrt{5}\cdot F_m},\frac{\sqrt{5}\cdot F_m}{L_m}\right]_m+1}{\left[\frac{L_n}{\sqrt{5}\cdot F_n},\frac{\sqrt{5}\cdot F_n}{L_n}\right]_n+\left[\frac{L_m}{\sqrt{5}\cdot F_m},\frac{\sqrt{5}\cdot F_m}{L_m}\right]_m}$$

$$=\begin{cases}\dfrac{\dfrac{L_n}{\sqrt{5}\cdot F_n}\cdot\dfrac{L_m}{\sqrt{5}\cdot F_m}+1}{\dfrac{L_n}{\sqrt{5}\cdot F_n}+\dfrac{L_m}{\sqrt{5}\cdot F_m}}, \text{if } n \text{ is odd and } m \text{ is odd}\\ \dfrac{\dfrac{L_n}{\sqrt{5}\cdot F_n}\cdot\dfrac{\sqrt{5}\cdot F_m}{L_m}+1}{\dfrac{L_n}{\sqrt{5}\cdot F_n}+\dfrac{\sqrt{5}\cdot F_m}{L_m}}, \text{if } n \text{ is odd and } m \text{ is even}\\ \dfrac{\dfrac{\sqrt{5}\cdot F_n}{L_n}\cdot\dfrac{L_m}{\sqrt{5}\cdot F_m}+1}{\dfrac{\sqrt{5}\cdot F_n}{L_n}+\dfrac{L_m}{\sqrt{5}\cdot F_m}}, \text{if } n \text{ is even and } m \text{ is odd}\\ \dfrac{\dfrac{\sqrt{5}\cdot F_n}{L_n}\cdot\dfrac{\sqrt{5}\cdot F_m}{L_m}+1}{\dfrac{\sqrt{5}\cdot F_n}{L_n}+\dfrac{\sqrt{5}\cdot F_m}{L_m}}, \text{if } n \text{ is even and } m \text{ is even}\end{cases}$$

$$=\begin{cases}\dfrac{\dfrac{a^n+b^n}{a^n-b^n}\cdot\dfrac{a^m+b^m}{a^m-b^m}+1}{\dfrac{a^n+b^n}{a^n-b^n}+\dfrac{a^m+b^m}{a^m-b^m}}, \text{if } n \text{ is odd and } m \text{ is odd}\\ \dfrac{\dfrac{a^n+b^n}{a^n-b^n}\cdot\dfrac{a^m-b^m}{a^m+b^m}+1}{\dfrac{a^n+b^n}{a^n-b^n}+\dfrac{a^m-b^m}{a^m+b^m}}, \text{if } n \text{ is odd and } m \text{ is even}\\ \dfrac{\dfrac{a^n-b^n}{a^n+b^n}\cdot\dfrac{a^m+b^m}{a^m-b^m}+1}{\dfrac{a^n-b^n}{a^n+b^n}+\dfrac{a^m+b^m}{a^m-b^m}}, \text{if } n \text{ is even and } m \text{ is odd}\\ \dfrac{\dfrac{a^n-b^n}{a^n+b^n}\cdot\dfrac{a^m-b^m}{a^m+b^m}+1}{\dfrac{a^n-b^n}{a^n+b^n}+\dfrac{a^m-b^m}{a^m+b^m}}, \text{if } n \text{ is even and } m \text{ is even}\end{cases}$$

$$=\begin{cases}\dfrac{(a^n+b^n)\cdot(a^m+b^m)+(a^n-b^n)\cdot(a^m-b^m)}{(a^n+b^n)\cdot(a^m-b^m)+(a^n-b^n)\cdot(a^m+b^m)}, \text{if } n \text{ is odd and } m \text{ is odd}\\ \dfrac{(a^n+b^n)\cdot(a^m-b^m)+(a^n-b^n)\cdot(a^m+b^m)}{(a^n+b^n)\cdot(a^m+b^m)+(a^n-b^n)\cdot(a^m-b^m)}, \text{if } n \text{ is odd and } m \text{ is even}\\ \dfrac{(a^n-b^n)\cdot(a^m+b^m)+(a^n+b^n)\cdot(a^m-b^m)}{(a^n-b^n)\cdot(a^m-b^m)+(a^n+b^n)\cdot(a^m+b^m)}, \text{if } n \text{ is even and } m \text{ is odd}\\ \dfrac{(a^n-b^n)\cdot(a^m-b^m)+(a^n+b^n)\cdot(a^m+b^m)}{(a^n-b^n)\cdot(a^m+b^m)+(a^n+b^n)\cdot(a^m-b^m)}, \text{if } n \text{ is even and } m \text{ is even}\end{cases}$$

$$=\begin{cases} \dfrac{a^{n+m}+b^{n+m}}{a^{n+m}-b^{n+m}}, \text{if } n \text{ is odd and } m \text{ is odd} \\ \dfrac{a^{n+m}-b^{n+m}}{a^{n+m}+b^{n+m}}, \text{if } n \text{ is odd and } m \text{ is even} \\ \dfrac{a^{n+m}-b^{n+m}}{a^{n+m}+b^{n+m}}, \text{if } n \text{ is even and } m \text{ is odd} \\ \dfrac{a^{n+m}+b^{n+m}}{a^{n+m}-b^{n+m}}, \text{if } n \text{ is even and } m \text{ is even} \end{cases}$$

$$=\begin{cases} \dfrac{L_{n+m}}{\sqrt{5}\cdot F_{n+m}}, \text{if } n \text{ is odd and } m \text{ is odd} \\ \dfrac{\sqrt{5}\cdot F_{n+m}}{L_{n+m}}, \text{if } n \text{ is odd and } m \text{ is even} \\ \dfrac{\sqrt{5}\cdot F_{n+m}}{L_{n+m}}, \text{if } n \text{ is even and } m \text{ is odd} \\ \dfrac{L_{n+m}}{\sqrt{5}\cdot F_{n+m}}, \text{if } n \text{ is even and } m \text{ is even} \end{cases}$$

$$=\begin{cases} \dfrac{L_{n+m}}{\sqrt{5}\cdot F_{n+m}}, \text{if } n+m \text{ is even} \\ \dfrac{\sqrt{5}\cdot F_{n+m}}{L_{n+m}}, \text{if } n+m \text{ is odd} \end{cases}$$

$$=\left[\frac{\sqrt{5}\cdot F_{n+m}}{L_{n+m}}, \frac{L_{n+m}}{\sqrt{5}\cdot F_{n+m}}\right]_{n+m};$$

therefore, we obtain again the equality (B.6).

***Otherwise*:** According to hypothesis and equalities (2.4) and (B.5),

(47) $$\text{cth}(x+y)=\frac{1}{\text{th}(x+y)}=\frac{1}{\left[\dfrac{L_{n+m}}{\sqrt{5}\cdot F_{n+m}}, \dfrac{\sqrt{5}\cdot F_{n+m}}{L_{n+m}}\right]_{n+m}}$$

$$=\begin{cases} \dfrac{1}{\dfrac{L_{n+m}}{\sqrt{5}\cdot F_{n+m}}}, \text{if } n+m \text{ is odd} \\ \dfrac{1}{\dfrac{\sqrt{5}\cdot F_{n+m}}{L_{n+m}}}, \text{if } n+m \text{ is even} \end{cases}$$

$$=\begin{cases}\dfrac{\sqrt{5}\cdot F_{n+m}}{L_{n+m}}, \text{if } n+m \text{ is odd}\\ \dfrac{L_{n+m}}{\sqrt{5}\cdot F_{n+m}}, \text{if } n+m \text{ is even}\end{cases}$$

$$=\left[\frac{\sqrt{5}\cdot F_{n+m}}{L_{n+m}}, \frac{L_{n+m}}{\sqrt{5}\cdot F_{n+m}}\right]_{n+m};$$

therefore, we obtain again the equality (B.6).

13) According to hypothesis and equalities (2.5) and (A.4),

$$\textbf{(48)}\,\mathrm{sch}(x+y)=\frac{1}{\mathrm{ch}(x+y)}=\frac{2}{\left[\sqrt{5}\cdot F_{n+m}, L_{n+m}\right]_{n+m}}$$

$$=\begin{cases}\dfrac{2}{\sqrt{5}\cdot F_{n+m}}, \text{if } n+m \text{ is odd}\\ \dfrac{2}{L_{n+m}} \quad, \text{if } n+m \text{ is even}\end{cases}$$

$$=\left[\frac{2}{\sqrt{5}\cdot F_{n+m}}, \frac{2}{L_{n+m}}\right]_{n+m};$$

which shows that the equality (B.7) holds.

***Otherwise*:** According to hypothesis and equalities (2.5′), (16) and (17),

$$\textbf{(49)}\,\mathrm{sch}(x+y)=\frac{2}{e^{x+y}+e^{-x-y}}=\frac{2}{e^{n\cdot\ln a+m\cdot\ln a}+e^{-n\cdot\ln a-m\cdot\ln a}}=\frac{2}{e^{\ln a^{n+m}}+e^{-\ln a^{n+m}}}$$

$$=\frac{2}{a^{n+m}+a^{-n-m}}=\frac{2}{a^{n+m}+\left(\dfrac{1}{a}\right)^{n+m}}=\frac{2}{a^{n+m}+(-b)^{n+m}}$$

$$=\begin{cases}\dfrac{2}{a^{n+m}-b^{n+m}}, \text{if } n+m \text{ is odd}\\ \dfrac{2}{a^{n+m}+b^{n+m}}, \text{if } n+m \text{ is even}\end{cases}$$

$$=\begin{cases}\dfrac{2}{\sqrt{5}\cdot F_{n+m}}, \text{if } n+m \text{ is odd}\\ \dfrac{2}{L_{n+m}} \quad, \text{if } n+m \text{ is even}\end{cases}$$

$$=\left[\frac{2}{\sqrt{5}\cdot F_{n+m}}, \frac{2}{L_{n+m}}\right]_{n+m};$$

therefore, we obtain again the equality (B.7).

14) According to hypothesis and equalities (2.6) and (A.3),

$$\textbf{(50)}\mathrm{csh}(x+y)=\frac{1}{\mathrm{sh}(x+y)}=\frac{2}{\left[L_{n+m},\sqrt{5}\cdot F_{n+m}\right]_{n+m}}$$

$$=\begin{cases}\dfrac{2}{L_{n+m}} & \text{, if } n+m \text{ is odd}\\ \dfrac{2}{\sqrt{5}\cdot F_{n+m}} & \text{, if } n+m \text{ is even}\end{cases}$$

$$=\left[\frac{2}{L_{n+m}},\frac{2}{\sqrt{5}\cdot F_{n+m}}\right]_{n+m};$$

which shows that the equality (B.8) holds.

Otherwise**:** According to hypothesis and equalities (2.6′), (16) and (17),

$$\textbf{(51)}\mathrm{csh}(x+y)=\frac{2}{e^{x+y}-e^{-x-y}}=\frac{2}{e^{n\cdot\ln a+m\cdot\ln a}-e^{-n\cdot\ln a-m\cdot\ln a}}=\frac{2}{e^{\ln a^{n+m}}-e^{-\ln a^{n+m}}}$$

$$=\frac{2}{a^{n+m}-a^{-n-m}}=\frac{2}{a^{n+m}-\left(\frac{1}{a}\right)^{n+m}}=\frac{2}{a^{n+m}-(-b)^{n+m}}$$

$$=\begin{cases}\dfrac{2}{a^{n+m}+b^{n+m}} & \text{, if } n+m \text{ is odd}\\ \dfrac{2}{a^{n+m}-b^{n+m}} & \text{, if } n+m \text{ is even}\end{cases}$$

$$=\begin{cases}\dfrac{2}{L_{n+m}} & \text{, if } n+m \text{ is odd}\\ \dfrac{2}{\sqrt{5}\cdot F_{n+m}} & \text{, if } n+m \text{ is even}\end{cases}$$

$$=\left[\frac{2}{L_{n+m}},\frac{2}{\sqrt{5}\cdot F_{n+m}}\right]_{n+m};$$

therefore, we obtain again the equality (B.8).

15) According to hypothesis and equalities (2.3), (A.5) and (A.6),

$$\textbf{(52)}\mathrm{th}(x\text{-}y)=\frac{\mathrm{sh}(x-y)}{\mathrm{ch}(x-y)}=\frac{\left[L_{n-m},\sqrt{5}\cdot F_{n-m}\right]_{n-m}}{\left[\sqrt{5}\cdot F_{n-m},L_{n-m}\right]_{n-m}}$$

$$=\begin{cases}\dfrac{L_{n-m}}{\sqrt{5}\cdot F_{n-m}} & \text{, if } n-m \text{ is odd}\\ \dfrac{\sqrt{5}\cdot F_{n-m}}{L_{n-m}} & \text{, if } n-m \text{ is even}\end{cases}$$

$$=\left[\frac{L_{n-m}}{\sqrt{5}\cdot F_{n-m}},\frac{\sqrt{5}\cdot F_{n-m}}{L_{n-m}}\right]_{n-m};$$

which shows that the equality (B.9) holds.

***Otherwise*:** According to hypothesis and equalities (2.3′), (16) and (17),

$$\textbf{(53)}\,\mathrm{th}(x\text{-}y)=\frac{e^{x-y}-e^{-x+y}}{e^{x-y}+e^{-x+y}}=\frac{e^{n\cdot\ln a-m\cdot\ln a}-e^{-n\cdot\ln a+m\cdot\ln a}}{e^{n\cdot\ln a-m\cdot\ln a}+e^{-n\cdot\ln a+m\cdot\ln a}}=\frac{e^{\ln a^{n-m}}-e^{-\ln a^{n-m}}}{e^{\ln a^{n-m}}+e^{-\ln a^{n-m}}}$$

$$=\frac{a^{n-m}-a^{-n+m}}{a^{n-m}+a^{-n+m}}=\frac{a^{n-m}-\left(\frac{1}{a}\right)^{n-m}}{a^{n-m}+\left(\frac{1}{a}\right)^{n-m}}=\frac{a^{n-m}-(-b)^{n-m}}{a^{n-m}+(-b)^{n-m}}$$

$$=\begin{cases}\dfrac{a^{n-m}+b^{n-m}}{a^{n-m}-b^{n-m}}, \text{if } n-m \text{ is odd}\\ \dfrac{a^{n-m}-b^{n-m}}{a^{n-m}+b^{n-m}}, \text{if } n-m \text{ is even}\end{cases}$$

$$=\begin{cases}\dfrac{L_{n-m}}{\sqrt{5}\cdot F_{n-m}}, \text{if } n-m \text{ is odd}\\ \dfrac{\sqrt{5}\cdot F_{n-m}}{L_{n-m}}, \text{if } n-m \text{ is even}\end{cases}$$

$$=\left[\frac{L_{n-m}}{\sqrt{5}\cdot F_{n-m}},\frac{\sqrt{5}\cdot F_{n-m}}{L_{n-m}}\right]_{n-m};$$

therefore, we obtain again the equality (B.9).

***Otherwise*:** According to hypothesis and equalities (3.7), (B.1), (16) and (17),

$$\textbf{(54)}\,\mathrm{th}(x\text{-}y)=\frac{\mathrm{th}x-\mathrm{th}y}{1-\mathrm{th}x\cdot\mathrm{th}y}=\frac{\left[\frac{L_n}{\sqrt{5}\cdot F_n},\frac{\sqrt{5}\cdot F_n}{L_n}\right]_n-\left[\frac{L_m}{\sqrt{5}\cdot F_m},\frac{\sqrt{5}\cdot F_m}{L_m}\right]_m}{1-\left[\frac{L_n}{\sqrt{5}\cdot F_n},\frac{\sqrt{5}\cdot F_n}{L_n}\right]_n\cdot\left[\frac{L_m}{\sqrt{5}\cdot F_m},\frac{\sqrt{5}\cdot F_m}{L_m}\right]_m}$$

$$
= \begin{cases}
\dfrac{\dfrac{L_n}{\sqrt{5}\cdot F_n} - \dfrac{L_m}{\sqrt{5}\cdot F_m}}{1 - \dfrac{L_n}{\sqrt{5}\cdot F_n}\cdot\dfrac{L_m}{\sqrt{5}\cdot F_m}}, \text{if } \ n \text{ is odd and } m \text{ is odd} \\[2ex]
\dfrac{\dfrac{L_n}{\sqrt{5}\cdot F_n} - \dfrac{\sqrt{5}\cdot F_m}{L_m}}{1 - \dfrac{L_n}{\sqrt{5}\cdot F_n}\cdot\dfrac{\sqrt{5}\cdot F_m}{L_m}}, \text{if } n \text{ is odd and } m \text{ is even} \\[2ex]
\dfrac{\dfrac{\sqrt{5}\cdot F_n}{L_n} - \dfrac{L_m}{\sqrt{5}\cdot F_m}}{1 - \dfrac{\sqrt{5}\cdot F_n}{L_n}\cdot\dfrac{L_m}{\sqrt{5}\cdot F_m}}, \text{if } n \text{ is even and } m \text{ is odd} \\[2ex]
\dfrac{\dfrac{\sqrt{5}\cdot F_n}{L_n} - \dfrac{\sqrt{5}\cdot F_m}{L_m}}{1 - \dfrac{\sqrt{5}\cdot F_n}{L_n}\cdot\dfrac{\sqrt{5}\cdot F_m}{L_m}}, \text{if } n \text{ is even and } m \text{ is even}
\end{cases}
$$

$$=\begin{cases}\dfrac{\dfrac{a^n+b^n}{a^n-b^n}-\dfrac{a^m+b^m}{a^m-b^m}}{1-\dfrac{a^n+b^n}{a^n-b^n}\cdot\dfrac{a^m+b^m}{a^m-b^m}}, \text{if } n \text{ is odd and } m \text{ is odd}\\ \dfrac{\dfrac{a^n+b^n}{a^n-b^n}-\dfrac{a^m-b^m}{a^m+b^m}}{1-\dfrac{a^n+b^n}{a^n-b^n}\cdot\dfrac{a^m-b^m}{a^m+b^m}}, \text{if } n \text{ is odd and } m \text{ is even}\\ \dfrac{\dfrac{a^n-b^n}{a^n+b^n}-\dfrac{a^m+b^m}{a^m-b^m}}{1-\dfrac{a^n-b^n}{a^n+b^n}\cdot\dfrac{a^m+b^m}{a^m-b^m}}, \text{if } n \text{ is even and } m \text{ is odd}\\ \dfrac{\dfrac{a^n-b^n}{a^n+b^n}-\dfrac{a^m-b^m}{a^m+b^m}}{1-\dfrac{a^n-b^n}{a^n+b^n}\cdot\dfrac{a^m-b^m}{a^m+b^m}}, \text{if } n \text{ is even and } m \text{ is even}\end{cases}$$

$$=\begin{cases}\dfrac{(a^n+b^n)\cdot(a^m-b^m)-(a^n-b^n)\cdot(a^m+b^m)}{(a^n-b^n)\cdot(a^m-b^m)-(a^n+b^n)\cdot(a^m+b^m)}, \text{if } n \text{ is odd and } m \text{ is odd}\\ \dfrac{(a^n+b^n)\cdot(a^m+b^m)-(a^n-b^n)\cdot(a^m-b^m)}{(a^n-b^n)\cdot(a^m+b^m)-(a^n+b^n)\cdot(a^m-b^m)}, \text{if } n \text{ is odd and } m \text{ is even}\\ \dfrac{(a^n-b^n)\cdot(a^m-b^m)-(a^n+b^n)\cdot(a^m+b^m)}{(a^n+b^n)\cdot(a^m-b^m)-(a^n-b^n)\cdot(a^m+b^m)}, \text{if } n \text{ is even and } m \text{ is odd}\\ \dfrac{(a^n-b^n)\cdot(a^m+b^m)-(a^n+b^n)\cdot(a^m-b^m)}{(a^n+b^n)\cdot(a^m+b^m)-(a^n-b^n)\cdot(a^m-b^m)}, \text{if } n \text{ is even and } m \text{ is even}\end{cases}$$

$$=\begin{cases}\dfrac{a^n\cdot b^m-a^m\cdot b^n}{a^m\cdot b^n+a^n\cdot b^m}, \text{if } n \text{ is odd and } m \text{ is odd}\\ \dfrac{a^n\cdot b^m+a^m\cdot b^n}{a^m\cdot b^n-a^n\cdot b^m}, \text{if } n \text{ is odd and } m \text{ is even}\\ \dfrac{a^n\cdot b^m+a^m\cdot b^n}{a^m\cdot b^n-a^n\cdot b^m}, \text{if } n \text{ is even and } m \text{ is odd}\\ \dfrac{a^n\cdot b^m-a^m\cdot b^n}{a^m\cdot b^n+a^n\cdot b^m}, \text{if } n \text{ is even and } m \text{ is even}\end{cases}$$

$$=\begin{cases}\dfrac{\dfrac{a^n\cdot b^m-a^m\cdot b^n}{(-1)^m}}{\dfrac{a^m\cdot b^n+a^n\cdot b^m}{(-1)^m}}, \text{if n is odd and m is odd}\\[2ex]\dfrac{\dfrac{a^n\cdot b^m+a^m\cdot b^n}{(-1)^m}}{\dfrac{a^m\cdot b^n-a^n\cdot b^m}{(-1)^m}}, \text{if n is odd and m is even}\\[2ex]\dfrac{\dfrac{a^n\cdot b^m+a^m\cdot b^n}{(-1)^m}}{\dfrac{a^m\cdot b^n-a^n\cdot b^m}{(-1)^m}}, \text{if n is even and m is odd}\\[2ex]\dfrac{\dfrac{a^n\cdot b^m-a^m\cdot b^n}{(-1)^m}}{\dfrac{a^m\cdot b^n+a^n\cdot b^m}{(-1)^m}}, \text{if n is even and m is even}\end{cases}$$

$$=\begin{cases}\dfrac{a^{n-m}-b^{n-m}}{a^{n-m}+b^{n-m}}, \text{if n is odd and m is impar}\\[1ex]\dfrac{a^{n-m}+b^{n-m}}{a^{n-m}-b^{n-m}}, \text{if n is odd and m is even}\\[1ex]\dfrac{a^{n-m}+b^{n-m}}{a^{n-m}-b^{n-m}}, \text{if n is even and m is odd}\\[1ex]\dfrac{a^{n-m}-b^{n-m}}{a^{n-m}+b^{n-m}}, \text{if n is even and m is even}\end{cases}$$

$$=\begin{cases}\dfrac{a^{n-m}+b^{n-m}}{a^{n-m}-b^{n-m}}, \text{if n - m is odd}\\[1ex]\dfrac{a^{n-m}-b^{n-m}}{a^{n-m}+b^{n-m}}, \text{if n - m is even}\end{cases}$$

$$=\begin{cases}\dfrac{L_{n-m}}{\sqrt{5}\cdot F_{n-m}}, \text{if n - m is even}\\[1ex]\dfrac{\sqrt{5}\cdot F_{n-m}}{L_{n-m}}, \text{if n - m is odd}\end{cases}$$

$$=\left[\frac{L_{n-m}}{\sqrt{5}\cdot F_{n-m}},\frac{\sqrt{5}\cdot F_{n-m}}{L_{n-m}}\right]_{n-m};$$

therefore, we obtain again the equality (B.9).

16) According to hypothesis and equalities (2.4), (A.5) and (A.6),

$$\textbf{(55)}\text{cth(x-y)}=\frac{\text{ch}(x-y)}{\text{sh}(x-y)}=\frac{\left[\sqrt{5}\cdot F_{n-m},L_{n-m}\right]_{n-m}}{\left[L_{n-m},\sqrt{5}\cdot F_{n+m}\right]_{n-m}}$$

$$=\begin{cases}\dfrac{\sqrt{5}\cdot F_{n-m}}{L_{n-m}}, \text{if } n-m \text{ is odd}\\ \dfrac{L_{n-m}}{\sqrt{5}\cdot F_{n-m}}, \text{if } n-m \text{ is even}\end{cases}$$

$$=\left[\frac{\sqrt{5}\cdot F_{n-m}}{L_{n-m}}, \frac{L_{n-m}}{\sqrt{5}\cdot F_{n-m}}\right]_{n-m};$$

which shows that the equality (B.10) holds.

Otherwise**:** According to hypothesis and equalities (2.4′), (16) and (17),

$$\textbf{(56)}\text{cth(x-y)}=\frac{e^{x-y}+e^{-x+y}}{e^{x-y}-e^{-x+y}}=\frac{e^{n\cdot\ln a-m\cdot\ln a}+e^{-n\cdot\ln a+m\cdot\ln a}}{e^{n\cdot\ln a-m\cdot\ln a}-e^{-n\cdot\ln a+m\cdot\ln a}}=\frac{e^{\ln a^{n-m}}+e^{-\ln a^{n-m}}}{e^{\ln a^{n-m}}-e^{-\ln a^{n-m}}}$$

$$=\frac{a^{n-m}+a^{-n+m}}{a^{n-m}-a^{-n+m}}=\frac{a^{n-m}+\left(\frac{1}{a}\right)^{n-m}}{a^{n-m}-\left(\frac{1}{a}\right)^{n-m}}=\frac{a^{n-m}+(-b)^{n-m}}{a^{n-m}-(-b)^{n-m}}$$

$$=\begin{cases}\dfrac{a^{n-m}-b^{n-m}}{a^{n-m}+b^{n-m}}, \text{if } n-m \text{ is odd}\\ \dfrac{a^{n-m}-b^{n-m}}{a^{n-m}+b^{n-m}}, \text{if } n-m \text{ is even}\end{cases}$$

$$=\begin{cases}\dfrac{\sqrt{5}\cdot F_{n-m}}{L_{n-m}}, \text{if } n-m \text{ is odd}\\ \dfrac{L_{n-m}}{\sqrt{5}\cdot F_{n-m}}, \text{if } n-m \text{ is even}\end{cases}$$

$$=\left[\frac{\sqrt{5}\cdot F_{n-m}}{L_{n-m}}, \frac{L_{n-m}}{\sqrt{5}\cdot F_{n-m}}\right]_{n-m};$$

therefore, we obtain again the equality (B.10).

Otherwise**:** According to hypothesis and equalities (3.9), (B.1), (16) and (17),

$$\textbf{(57)}\text{cth(x-y)}=\frac{\text{cthx}\cdot\text{cthy}-1}{-\text{cthx}+\text{cthy}}=\frac{\left[\frac{L_n}{\sqrt{5}\cdot F_n}, \frac{\sqrt{5}\cdot F_n}{L_n}\right]_n\cdot\left[\frac{L_m}{\sqrt{5}\cdot F_m}, \frac{\sqrt{5}\cdot F_m}{L_m}\right]_m-1}{-\left[\frac{L_n}{\sqrt{5}\cdot F_n}, \frac{\sqrt{5}\cdot F_n}{L_n}\right]_n+\left[\frac{L_m}{\sqrt{5}\cdot F_m}, \frac{\sqrt{5}\cdot F_m}{L_m}\right]_m}$$

$$=\begin{cases}\dfrac{\dfrac{L_n}{\sqrt{5}\cdot F_n}\cdot\dfrac{L_m}{\sqrt{5}\cdot F_m}-1}{-\dfrac{L_n}{\sqrt{5}\cdot F_n}+\dfrac{L_m}{\sqrt{5}\cdot F_m}}, \text{if n is odd and m is odd}\\[2ex] \dfrac{\dfrac{L_n}{\sqrt{5}\cdot F_n}\cdot\dfrac{\sqrt{5}\cdot F_m}{L_m}-1}{-\dfrac{L_n}{\sqrt{5}\cdot F_n}+\dfrac{\sqrt{5}\cdot F_m}{L_m}}, \text{if n is odd and m is even}\\[2ex] \dfrac{\dfrac{\sqrt{5}\cdot F_n}{L_n}\cdot\dfrac{L_m}{\sqrt{5}\cdot F_m}-1}{-\dfrac{\sqrt{5}\cdot F_n}{L_n}+\dfrac{L_m}{\sqrt{5}\cdot F_m}}, \text{if n is even and m is odd}\\[2ex] \dfrac{\dfrac{\sqrt{5}\cdot F_n}{L_n}\cdot\dfrac{\sqrt{5}\cdot F_m}{L_m}-1}{-\dfrac{\sqrt{5}\cdot F_n}{L_n}+\dfrac{\sqrt{5}\cdot F_m}{L_m}}, \text{if n is even and m is even}\end{cases}$$

$$=\begin{cases}\dfrac{\dfrac{a^n+b^n}{a^n-b^n}\cdot\dfrac{a^m+b^m}{a^m-b^m}-1}{-\dfrac{a^n+b^n}{a^n-b^n}+\dfrac{a^m+b^m}{a^m-b^m}}, \text{if n is odd and m is odd}\\[2ex] \dfrac{\dfrac{a^n+b^n}{a^n-b^n}\cdot\dfrac{a^m-b^m}{a^m+b^m}-1}{-\dfrac{a^n+b^n}{a^n-b^n}+\dfrac{a^m-b^m}{a^m+b^m}}, \text{if n is odd and m is even}\\[2ex] \dfrac{\dfrac{a^n-b^n}{a^n+b^n}\cdot\dfrac{a^m+b^m}{a^m-b^m}-1}{-\dfrac{a^n-b^n}{a^n+b^n}+\dfrac{a^m+b^m}{a^m-b^m}}, \text{if n is even and m is odd}\\[2ex] \dfrac{\dfrac{a^n-b^n}{a^n+b^n}\cdot\dfrac{a^m-b^m}{a^m+b^m}-1}{-\dfrac{a^n-b^n}{a^n+b^n}+\dfrac{a^m-b^m}{a^m+b^m}}, \text{if n is even and m is even}\end{cases}$$

$$=\begin{cases}\dfrac{(a^n+b^n)\cdot(a^m+b^m)-(a^n-b^n)\cdot(a^m-b^m)}{-(a^n+b^n)\cdot(a^m-b^m)+(a^n-b^n)\cdot(a^m+b^m)}, \text{if n is odd and m is odd}\\[2ex] \dfrac{(a^n+b^n)\cdot(a^m-b^m)-(a^n-b^n)\cdot(a^m+b^m)}{-(a^n+b^n)\cdot(a^m+b^m)+(a^n-b^n)\cdot(a^m-b^m)}, \text{if n is odd and m is even}\\[2ex] \dfrac{(a^n-b^n)\cdot(a^m+b^m)-(a^n+b^n)\cdot(a^m-b^m)}{-(a^n-b^n)\cdot(a^m-b^m)+(a^n+b^n)\cdot(a^m+b^m)}, \text{if n is even and m is odd}\\[2ex] \dfrac{(a^n-b^n)\cdot(a^m-b^m)-(a^n+b^n)\cdot(a^m+b^m)}{-(a^n-b^n)\cdot(a^m+b^m)+(a^n+b^n)\cdot(a^m-b^m)}, \text{if n is even and m is even}\end{cases}$$

$$=\begin{cases}\dfrac{a^n\cdot b^m+a^m\cdot b^n}{a^n\cdot b^m-a^m\cdot b^n}, \text{if } \ n \text{ is odd and } m \text{ is odd}\\ \dfrac{a^n\cdot b^m-a^m\cdot b^n}{a^n\cdot b^m+a^m\cdot b^n}, \text{if } n \text{ is odd and } m \text{ is even}\\ \dfrac{a^n\cdot b^m-a^m\cdot b^n}{a^n\cdot b^m+a^m\cdot b^n}, \text{if } n \text{ is even and } m \text{ is odd}\\ \dfrac{a^n\cdot b^m+a^m\cdot b^n}{a^n\cdot b^m-a^m\cdot b^n}, \text{if } n \text{ is even and } m \text{ is even}\end{cases}$$

$$=\begin{cases}\dfrac{\frac{a^n\cdot b^m+a^m\cdot b^n}{(-1)^m}}{\frac{a^n\cdot b^m-a^m\cdot b^n}{(-1)^m}}, \text{if } \ n \text{ is odd and } m \text{ is odd}\\ \dfrac{\frac{a^n\cdot b^m-a^m\cdot b^n}{(-1)^m}}{\frac{a^n\cdot b^m+a^m\cdot b^n}{(-1)^m}}, \text{if } n \text{ is odd and } m \text{ is even}\\ \dfrac{\frac{a^n\cdot b^m-a^m\cdot b^n}{(-1)^m}}{\frac{a^n\cdot b^m+a^m\cdot b^n}{(-1)^m}}, \text{if } n \text{ is par and } m \text{ is odd}\\ \dfrac{\frac{a^n\cdot b^m+a^m\cdot b^n}{(-1)^m}}{\frac{a^n\cdot b^m-a^m\cdot b^n}{(-1)^m}}, \text{if } n \text{ is even and } m \text{ is even}\end{cases}$$

$$=\begin{cases}\dfrac{\frac{a^n\cdot b^m+a^m\cdot b^n}{a^m\cdot b^m}}{\frac{a^n\cdot b^m-a^m\cdot b^n}{a^m\cdot b^m}}, \text{if } \ n \text{ is odd and } m \text{ is odd}\\ \dfrac{\frac{a^n\cdot b^m-a^m\cdot b^n}{a^m\cdot b^m}}{\frac{a^n\cdot b^m+a^m\cdot b^n}{a^m\cdot b^m}}, \text{if } n \text{ is odd and } m \text{ is even}\\ \dfrac{\frac{a^n\cdot b^m-a^m\cdot b^n}{a^m\cdot b^m}}{\frac{a^n\cdot b^m+a^m\cdot b^n}{a^m\cdot b^m}}, \text{if } n \text{ is even and } m \text{ is odd}\\ \dfrac{\frac{a^n\cdot b^m+a^m\cdot b^n}{a^m\cdot b^m}}{\frac{a^n\cdot b^m-a^m\cdot b^n}{a^m\cdot b^m}}, \text{if } n \text{ is even and } m \text{ is even}\end{cases}$$

$$=\begin{cases} \dfrac{a^{n-m}+b^{n-m}}{a^{n-m}-b^{n-m}}, \text{if } \ n \text{ is odd and } m \text{ is odd} \\ \dfrac{a^{n-m}-b^{n-m}}{a^{n-m}+b^{n-m}}, \text{if } n \text{ is odd and } m \text{ is even} \\ \dfrac{a^{n-m}-b^{n-m}}{a^{n-m}+b^{n-m}}, \text{if } n \text{ is even and } m \text{ is odd} \\ \dfrac{a^{n-m}+b^{n-m}}{a^{n-m}-b^{n-m}}, \text{if } n \text{ is even and } m \text{ is even} \end{cases}$$

$$=\begin{cases} \dfrac{a^{n-m}-b^{n-m}}{a^{n-m}+b^{n-m}}, \text{if } n-m \text{ is odd} \\ \dfrac{a^{n-m}+b^{n-m}}{a^{n-m}-b^{n-m}}, \text{if } n-m \text{ is even} \end{cases}$$

$$=\begin{cases} \dfrac{L_{n-m}}{\sqrt{5}\cdot F_{n-m}}, \text{if } n-m \text{ is even} \\ \dfrac{\sqrt{5}\cdot F_{n-m}}{L_{n-m}}, \text{if } n-m \text{ is odd} \end{cases}$$

$$=\left[\frac{\sqrt{5}\cdot F_{n-m}}{L_{n-m}}, \frac{L_{n-m}}{\sqrt{5}\cdot F_{n-m}}\right]_{n-m};$$

therefore, we obtain again the equality (B.10).

***Otherwise*:** According to hypothesis and equalities (2.4) and (B.9),

$$\textbf{(58)}\,\mathrm{cth}(x\text{-}y)=\frac{1}{\mathrm{th}(x-y)}=\frac{1}{\left[\dfrac{L_{n-m}}{\sqrt{5}\cdot F_{n-m}}, \dfrac{\sqrt{5}\cdot F_{n-m}}{L_{n-m}}\right]_{n-m}}$$

$$=\begin{cases} \dfrac{1}{\dfrac{L_{n-m}}{\sqrt{5}\cdot F_{n-m}}}, \text{if } n-m \text{ is odd} \\ \dfrac{1}{\dfrac{\sqrt{5}\cdot F_{n-m}}{L_{n-m}}}, \text{if } n-m \text{ is even} \end{cases}$$

$$=\begin{cases} \dfrac{\sqrt{5}\cdot F_{n+m}}{L_{n+m}}, \text{if } n-m \text{ is odd} \\ \dfrac{L_{n-m}}{\sqrt{5}\cdot F_{n-m}}, \text{if } n-m \text{ is even} \end{cases}$$

$$=\left[\frac{\sqrt{5}\cdot F_{n-m}}{L_{n-m}}, \frac{L_{n-m}}{\sqrt{5}\cdot F_{n-m}}\right]_{n-m};$$

therefore, we obtain again the equality (B.10).

17) According to hypothesis and equalities (2.5) and (A.6),

$$\textbf{(59)}\mathrm{sch}(x\text{-}y)=\frac{1}{\mathrm{ch}(x-y)}=\frac{2}{\left[\sqrt{5}\cdot F_{n-m},L_{n-m}\right]_{n-m}}$$

$$=\begin{cases}\dfrac{2}{\sqrt{5}\cdot F_{n-m}}, \text{if } n-m \text{ is odd}\\ \dfrac{2}{L_{n-m}} \quad, \text{if } n-m \text{ is even}\end{cases}$$

$$=\left[\frac{2}{\sqrt{5}\cdot F_{n-m}},\frac{2}{L_{n-m}}\right]_{n-m};$$

which shows that the equality (B.11) holds.

***Otherwise*:** According to hypothesis and equalities (2.5′), (16) and (17),

$$\textbf{(60)}\mathrm{sch}(x\text{-}y)=\frac{2}{e^{x-y}+e^{-x+y}}=\frac{2}{e^{n\cdot\ln a-m\cdot\ln a}+e^{-n\cdot\ln a+m\cdot\ln a}}=\frac{2}{e^{\ln a^{n-m}}+e^{-\ln a^{n-m}}}$$

$$=\frac{2}{a^{n-m}+a^{-n+m}}=\frac{2}{a^{n-m}+\left(\frac{1}{a}\right)^{n-m}}=\frac{2}{a^{n-m}+(-b)^{n-m}}$$

$$=\begin{cases}\dfrac{2}{a^{n-m}-b^{n-m}}, \text{if } n-m \text{ is odd}\\ \dfrac{2}{a^{n-m}+b^{n-m}}, \text{if } n-m \text{ is even}\end{cases}$$

$$=\begin{cases}\dfrac{2}{\sqrt{5}\cdot F_{n-m}}, \text{if } n-m \text{ is odd}\\ \dfrac{2}{L_{n-m}} \quad, \text{if } n-m \text{ is even}\end{cases}$$

$$=\left[\frac{2}{\sqrt{5}\cdot F_{n-m}},\frac{2}{L_{n-m}}\right]_{n-m};$$

therefore, we obtain again the equality (B.11).

18) According to hypothesis and equalities (2.6) and (A.5),

$$\textbf{(61)}\mathrm{csh}(x\text{-}y)=\frac{1}{\mathrm{sh}(x-y)}=\frac{2}{\left[L_{n-m},\sqrt{5}\cdot F_{n-m}\right]_{n-m}}$$

$$=\begin{cases}\dfrac{2}{L_{n-m}} \quad, \text{if } n-m \text{ is odd}\\ \dfrac{2}{\sqrt{5}\cdot F_{n-m}}, \text{if } n-m \text{ is even}\end{cases}$$

$$=\left[\frac{2}{L_{n-m}},\frac{2}{\sqrt{5}\cdot F_{n-m}}\right]_{n-m};$$

which shows that the equality (B.12) holds.

***Otherwise*:** According to hypothesis and equalities (2.6′), (16) and (17),

$$\textbf{(62)}\operatorname{csh}(x\text{-}y)=\frac{2}{e^{x-y}-e^{-x+y}}=\frac{2}{e^{n\cdot\ln a-m\cdot\ln a}-e^{-n\cdot\ln a+m\cdot\ln a}}=\frac{2}{e^{\ln a^{n-m}}-e^{-\ln a^{n-m}}}$$

$$=\frac{2}{a^{n-m}-a^{-n+m}}=\frac{2}{a^{n-m}-\left(\frac{1}{a}\right)^{n-m}}=\frac{2}{a^{n-m}-(-b)^{n-m}}$$

$$=\begin{cases}\dfrac{2}{a^{n-m}+b^{n-m}}, \text{if } n-m \text{ is odd}\\ \dfrac{2}{a^{n-m}-b^{n-m}}, \text{if } n-m \text{ is even}\end{cases}$$

$$=\begin{cases}\dfrac{2}{L_{n-m}}, \text{if } n-m \text{ is odd}\\ \dfrac{2}{\sqrt{5}\cdot F_{n-m}}, \text{if } n-m \text{ is even}\end{cases}$$

$$=\left[\frac{2}{L_{n-m}},\frac{2}{\sqrt{5}\cdot F_{n-m}}\right]_{n-m};$$

therefore, we obtain again the equality (B.12). □

Now we can obtain other relationships between the terms of our sequences.

Corollary C: *For every m, n∈**N**, the following equalities hold*:

19) $L_n^2\text{-}5\cdot F_n^2=4\cdot(-1)^n$; (C.1)

20) $\begin{cases}F_{n+m}+F_{n-m}=[L_n\cdot F_m,F_n\cdot L_m]_m\\ L_{n+m}+L_{n-m}=[5\cdot F_n\cdot F_m,L_n\cdot L_m]_m\end{cases}$, (C.2)

in particular:

21) $\begin{cases}L_{n+1}+L_{n-1}=5\cdot F_n\\ F_{n+1}+F_{n-1}=L_n\end{cases}$; (C.2′)

22) $\begin{cases}L_{n+m}-L_{n-m}=[L_n\cdot L_m,5\cdot F_n\cdot F_m]_m\\ F_{n+m}-F_{n-m}=[F_n\cdot L_m,L_n\cdot F_m]_m\end{cases}$, (C.3)

in particular:

23) $\begin{cases}F_{n+1}-F_{n-1}=F_n\\ L_{n+1}-L_{n-1}=L_n\end{cases}$; (C.3′)

24) $\begin{cases}2\cdot L_{n+m}=L_n\cdot L_m+5\cdot F_n\cdot F_m\\ 2\cdot F_{n+m}=F_n\cdot L_m+L_n\cdot F_m\end{cases}$, (C.4)

in particular:

25) $\begin{cases} 2\cdot F_{n+1} = F_n + L_n \\ 2\cdot L_{n+1} = L_n + 5\cdot F_n \end{cases}$; (C.4′)

26) $\begin{cases} 2\cdot L_{n-m} = (-1)^{m-1}\cdot(5\cdot F_n\cdot F_m - L_n\cdot L_m) \\ 2\cdot F_{n-m} = (-1)^{m-1}\cdot(L_n\cdot F_m - F_n\cdot L_m) \end{cases}$, (C.5)

in particular:

27) $\begin{cases} 2\cdot F_{n-1} = L_n - F_n \\ 2\cdot L_{n-1} = 5\cdot F_n - L_n \end{cases}$; (C.5′)

28) $\begin{cases} F_{n+m}\cdot F_{n-m} - F_n^2 = (-1)^{n+m+1}\cdot F_m^2 \\ L_{n+m}\cdot L_{n-m} - L_n^2 = (-1)^{n+m}\cdot L_m^2 - 4\cdot(-1)^n \end{cases}$, (C.6)

in particular:

29) $\begin{cases} F_{n+1}\cdot F_{n-1} - F_n^2 = (-1)^n \\ L_{n+1}\cdot L_{n-1} - L_n^2 = 5\cdot(-1)^{n+1} \end{cases}$; (C.6′)

30) $\begin{cases} F_{n+2}\cdot F_{n-2} - F_n^2 = (-1)^{n+1} \\ L_{n+2}\cdot L_{n-2} - L_n^2 = 5\cdot(-1)^n \end{cases}$. (C.6″)

***Proof*: 19)** According to the equality (3.1), for every $x\in\mathbf{R}$,

$ch^2x - sh^2x = 1$,

which, according to the equalities (A.1) and (A.2), becomes:

(63)$([\sqrt{5}\cdot F_n, L_n]_n)^2 - ([L_n, \sqrt{5}\cdot F_n]_n)^2 = 4$.

From the equality (63) we obtain that:

$$\begin{cases} 5\cdot F_n^2 - L_n^2 = 4, \text{if } n \text{ is odd} \\ L_n^2 - 5\cdot F_n^2 = 4, \text{if } n \text{ is even} \end{cases};$$

which shows that equality (C.1) holds.

***Otherwise*:** From the equalities (9), (10) and (14), it follows that, for every $n\in\mathbf{N}$:

$L_n^2 - 5\cdot F_n^2 = (a^n+b^n)^2 - (a^n-b^n)^2 = 4\cdot a^n\cdot b^n$

$= 4\cdot(-1)^n$; since:

$a\cdot b = -1$.

20) + 21) From the equalities (4.31) and (4.63), it follows that, for every $x, y\in\mathbf{R}$:

(64)$sh(x+y) + sh(x-y) = 2\cdot shx\cdot chy$,

respectively

(65)$ch(x+y) + ch(x-y) = 2\cdot chx\cdot chy$.

Now, from the equalities (64), (65), (A.1), (A.2), (A.3), (A.4), (A.5) and (A.6), with the notations used above, it follows that, for every $m, n\in\mathbf{N}$:

(66) $\frac{1}{2}\cdot[L_{n+m}, \sqrt{5}\cdot F_{n+m}]_{n+m} + \frac{1}{2}\cdot[L_{n-m}, \sqrt{5}\cdot F_{n-m}]_{n-m} = 2\cdot\frac{1}{2}\cdot[L_n, \sqrt{5}\cdot F_n]_n\cdot\frac{1}{2}\cdot[\sqrt{5}\cdot F_m, L_m]_m$,

respectively:

(67) $\frac{1}{2}\cdot[\sqrt{5}\cdot F_{n+m},L_{n+m}]_{n+m}+\frac{1}{2}\cdot[\sqrt{5}\cdot F_{n-m},L_{n-m}]_{n-m}=2\cdot\frac{1}{2}\cdot[\sqrt{5}\cdot F_n,L_n]_n\cdot\frac{1}{2}\cdot[\sqrt{5}\cdot F_m,L_m]_m$,

that is:

(68) $[L_{n+m},\sqrt{5}\cdot F_{n+m}]_{n+m}+[L_{n-m},\sqrt{5}\cdot F_{n-m}]_{n-m}=[L_n,\sqrt{5}\cdot F_n]_n\cdot[\sqrt{5}\cdot F_m,L_m]_m$,

respectively:

(69) $[\sqrt{5}\cdot F_{n+m},L_{n+m}]_{n+m}+[\sqrt{5}\cdot F_{n-m},L_{n-m}]_{n-m}=[\sqrt{5}\cdot F_n,L_n]_n\cdot[\sqrt{5}\cdot F_m,L_m]_m$.

From the equalities (68), (69) and (∗), obtain that:

$$\textbf{(70)}\begin{cases}\sqrt{5}\cdot F_{n+m}+\sqrt{5}\cdot F_{n-m}=L_n\cdot\sqrt{5}\cdot F_m, \text{if n is odd and m is odd}\\ L_{n+m}+L_{n-m}=L_n\cdot L_m \quad , \text{if n is odd and m is even}\\ L_{n+m}+L_{n-m}=\sqrt{5}\cdot F_n\cdot\sqrt{5}\cdot F_m \quad , \text{if n is even and m is odd}\\ \sqrt{5}\cdot F_{n+m}+\sqrt{5}\cdot F_{n-m}=\sqrt{5}\cdot F_n\cdot L_m, \text{if n is even and m is even}\end{cases},$$

respectively:

$$\textbf{(71)}\begin{cases}L_{n+m}+L_{n-m}=\sqrt{5}\cdot F_n\cdot\sqrt{5}\cdot F_m \quad , \text{if n is odd and m is odd}\\ \sqrt{5}\cdot F_{n+m}+\sqrt{5}\cdot F_{n-m}=\sqrt{5}\cdot F_n\cdot L_m, \text{if n is odd and m is even}\\ \sqrt{5}\cdot F_{n+m}+\sqrt{5}\cdot F_{n-m}=L_n\cdot\sqrt{5}\cdot F_m, \text{if n is even and m is odd}\\ L_{n+m}+L_{n-m}=L_n\cdot L_m \quad , \text{if n is even and m is even}\end{cases}$$

that is:

$$\textbf{(72)}\begin{cases}F_{n+m}+F_{n-m}=L_n\cdot F_m \quad , \text{if n is odd and m is odd}\\ L_{n+m}+L_{n-m}=L_n\cdot L_m \quad , \text{if n is odd and m is even}\\ L_{n+m}+L_{n-m}=5\cdot F_n\cdot F_m, \text{if n is even and m is odd}\\ F_{n+m}+F_{n-m}=F_n\cdot L_m \quad , \text{if n is even and m is even}\end{cases},$$

respectively

$$\textbf{(73)}\begin{cases}L_{n+m}+L_{n-m}=5\cdot F_n\cdot F_m, \text{if n is odd and m is odd}\\ F_{n+m}+F_{n-m}=F_n\cdot L_m \quad , \text{if n is odd and m is even}\\ F_{n+m}+F_{n-m}=L_n\cdot F_m \quad , \text{if n is even and m is odd}\\ L_{n+m}+L_{n-m}=L_n\cdot L_m \quad , \text{if n is even and m is even}\end{cases}.$$

From the equalities (72) and (73), obtain that:

$$\textbf{(74)}\begin{cases}L_{n+m}+L_{n-m}=5\cdot F_n\cdot F_m, \text{if m is odd}\\ F_{n+m}+F_{n-m}=F_n\cdot L_m \quad , \text{if m is even}\\ F_{n+m}+F_{n-m}=L_n\cdot F_m \quad , \text{if m is odd}\\ L_{n+m}+L_{n-m}=L_n\cdot L_m \quad , \text{if m is even}\end{cases},$$

that is:

$$\textbf{(75)}\begin{cases}L_{n+m}+L_{n-m}=[5\cdot F_n\cdot F_m, L_n\cdot L_m]_m\\ F_{n+m}+F_{n-m}=[L_n\cdot F_m, F_n\cdot L_m]_m\end{cases}.$$

Therefore, the equalities (C.2) hold. For:

m=1,

from the equalities (C.2), obtain that:

$$\textbf{(76)}\begin{cases} L_{n+1}+L_{n-1}=[5\cdot F_n\cdot F_1, L_n\cdot L_1]_1 \\ F_{n+1}+F_{n-1}=[L_n\cdot F_1, F_n\cdot L_1]_1 \end{cases}.$$

Because:

$F_1=L_1=1$,

from the equalities (76) and (∗), obtain the equalities (C.2′):

$$\textbf{(77)}\begin{cases} L_{n+1}+L_{n-1}=5\cdot F_n \\ F_{n+1}+F_{n-1}=L_n \end{cases}.$$

***Otherwise*:** From the equalities (9), (10) and (14), it follows that, for every m, n$\in$**N**:

$$\textbf{(78)}F_{n+m}+F_{n-m}=\frac{a^{n+m}-b^{n+m}}{a-b}+\frac{a^{n-m}-b^{n-m}}{a-b}=\frac{1}{a-b}\cdot\left[a^n\cdot a^m-b^n\cdot b^m+\frac{a^n}{a^m}-\frac{b^n}{b^m}\right]$$

$$=\frac{1}{a-b}\cdot\left[a^n\cdot a^m-b^n\cdot b^m+\frac{a^n\cdot b^m-a^m\cdot b^n}{(a\cdot b)^m}\right]$$

$$=\frac{1}{a-b}\cdot\left[a^n\cdot a^m-b^n\cdot b^m+\frac{a^n\cdot b^m-a^m\cdot b^n}{(-1)^m}\right]$$

$$=\begin{cases} \frac{1}{a-b}\cdot[a^n\cdot a^m-b^n\cdot b^m-a^n\cdot b^m+b^n\cdot a^m], \text{if m is odd} \\ \frac{1}{a-b}\cdot[a^n\cdot a^m-b^n\cdot b^m+a^n\cdot b^m-b^n\cdot a^m], \text{if m is even} \end{cases}$$

$$=\begin{cases} (a^n+b^n)\cdot\frac{a^m-b^m}{a-b}, \text{if m is odd} \\ \frac{a^n-b^n}{a-b}\cdot(a^m+b^m), \text{if m is even} \end{cases}$$

$$=\begin{cases} L_n\cdot F_m, \text{if m is odd} \\ F_n\cdot L_m, \text{if m is even} \end{cases}$$

$=[L_n\cdot F_m, F_n\cdot L_m]_m$;

respectively:

$$\textbf{(79)}L_{n+m}+L_{n-m}=a^{n+m}+b^{n+m}+a^{n-m}+b^{n-m}=a^n\cdot a^m+b^n\cdot b^m+\frac{a^n}{a^m}+\frac{b^n}{b^m}$$

$$=a^n\cdot a^m+b^n\cdot b^m+\frac{a^n\cdot b^m+a^m\cdot b^n}{(a\cdot b)^m}=a^n\cdot a^m+b^n\cdot b^m+\frac{a^n\cdot b^m+a^m\cdot b^n}{(-1)^m}$$

$$=\begin{cases} a^n\cdot a^m+b^n\cdot b^m-a^n\cdot b^m-b^n\cdot a^m, \text{if m is odd} \\ a^n\cdot a^m+b^n\cdot b^m+a^n\cdot b^m+b^n\cdot a^m, \text{if m is even} \end{cases}$$

$$=\begin{cases} (a^n-b^n)\cdot(a^m-b^m), \text{if m is odd} \\ (a^n+b^n)\cdot(a^m+b^m), \text{if m is even} \end{cases}$$

$$=\begin{cases}5\cdot F_n\cdot F_m, \text{if } m \text{ is odd}\\ L_n\cdot L_m \quad, \text{if } m \text{ is even}\end{cases}$$

$=[5\cdot F_n\cdot F_m, L_n\cdot L_m]_m$;

that is, we obtained, again, the equalities (C.2). Analogous, for:

m=1

obtain that:

(80) $F_{n+1}+F_{n-1}=\dfrac{a^{n+1}-b^{n+1}}{a-b}+\dfrac{a^{n-1}-b^{n-1}}{a-b}=\dfrac{1}{a-b}\cdot\left[a^n\cdot a-b^n\cdot b+\dfrac{a^n}{a}-\dfrac{b^n}{b}\right]$

$=\dfrac{1}{a-b}\cdot\left[a^n\cdot a-b^n\cdot b+\dfrac{a^n\cdot b-a\cdot b^n}{a\cdot b}\right]=\dfrac{1}{a-b}\cdot\left[a^n\cdot a-b^n\cdot b+\dfrac{a^n\cdot b-a\cdot b^n}{-1}\right]$

$=\dfrac{1}{a-b}\cdot[a^n\cdot a-b^n\cdot b-a^n\cdot b+b^n\cdot a]=a^n+b^n$

$=L_n$;

respectively:

(81) $L_{n+1}+L_{n-1}=a^{n+1}+b^{n+1}+a^{n-1}+b^{n-1}=a^n\cdot a+b^n\cdot b+\dfrac{a^n}{a}+\dfrac{b^n}{b}$

$=a^n\cdot a+b^n\cdot b+\dfrac{a^n\cdot b+a\cdot b^n}{a\cdot b}=a^n\cdot a+b^n\cdot b+\dfrac{a^n\cdot b+a\cdot b^n}{-1}$

$=a^n\cdot a+b^n\cdot b-a^n\cdot b-b^n\cdot a=(a^n-b^n)\cdot(a-b)$

$=5\cdot F_n$;

that is, we obtained, again, the equalities (C.2′).

22) + 23) From the equalities (4.32) and (4.34), it follows that, for every $x, y\in\mathbf{R}$:

(82) $sh(x+y)-sh(x-y)=2\cdot chx\cdot shy$,

respectively

(83) $ch(x+y)-ch(x-y)=2\cdot shx\cdot shy$.

Now, from the equalities (82), (83), (A.1), (A.2), (A.3), (A.4), (A.5) and (A.6), with the notations used above, it follows that, for every $m, n\in\mathbf{N}$:

(84) $\frac{1}{2}\cdot[L_{n+m},\sqrt{5}\cdot F_{n+m}]_{n+m}-\frac{1}{2}\cdot[L_{n-m},\sqrt{5}\cdot F_{n-m}]_{n-m}=2\cdot\frac{1}{2}\cdot[\sqrt{5}\cdot F_n,L_n]_n\cdot\frac{1}{2}\cdot[L_m,\sqrt{5}\cdot F_m]_m$,

respectively:

(85) $\frac{1}{2}\cdot[\sqrt{5}\cdot F_{n+m},L_{n+m}]_{n+m}-\frac{1}{2}\cdot[\sqrt{5}\cdot F_{n-m},L_{n-m}]_{n-m}=2\cdot\frac{1}{2}\cdot[L_n,\sqrt{5}\cdot F_n]_n\cdot\frac{1}{2}\cdot[L_m,\sqrt{5}\cdot F_m]_m$,

that is:

(86) $[L_{n+m},\sqrt{5}\cdot F_{n+m}]_{n+m}-[L_{n-m},\sqrt{5}\cdot F_{n-m}]_{n-m}=[\sqrt{5}\cdot F_n,L_n]_n\cdot[L_m,\sqrt{5}\cdot F_m]_m$,

respectively:

(87) $[\sqrt{5}\cdot F_{n+m},L_{n+m}]_{n+m}-[\sqrt{5}\cdot F_{n-m},L_{n-m}]_{n-m}=[L_n,\sqrt{5}\cdot F_n]_n\cdot[L_m,\sqrt{5}\cdot F_m]_m$.

From the equalities (86), (87) and (∗), we deduce that:

$$(88)\begin{cases}\sqrt{5}\cdot F_{n+m}-\sqrt{5}\cdot F_{n-m}=\sqrt{5}\cdot F_n\cdot L_m, \text{if n is odd and m is odd}\\ L_{n+m}-L_{n-m}=\sqrt{5}\cdot F_n\cdot\sqrt{5}\cdot F_m \quad, \text{if n is odd and m is even}\\ L_{n+m}-L_{n-m}=L_n\cdot L_m \quad, \text{if n is even and m is odd}\\ \sqrt{5}\cdot F_{n+m}-\sqrt{5}\cdot F_{n-m}=L_n\cdot\sqrt{5}\cdot F_m, \text{if n is even and m is even}\end{cases},$$

respectively:

$$(89)\begin{cases}L_{n+m}-L_{n-m}=L_n\cdot L_m \quad, \text{if n is odd and m is odd}\\ \sqrt{5}\cdot F_{n+m}-\sqrt{5}\cdot F_{n-m}=L_n\cdot\sqrt{5}\cdot F_m, \text{if n is odd and m is even}\\ \sqrt{5}\cdot F_{n+m}-\sqrt{5}\cdot F_{n-m}=\sqrt{5}\cdot F_n\cdot L_m, \text{if n is even and m is odd}\\ L_{n+m}-L_{n-m}=\sqrt{5}\cdot F_n\cdot\sqrt{5}\cdot F_m \quad, \text{if n is even and m is even}\end{cases}$$

that is:

$$(90)\begin{cases}F_{n+m}-F_{n-m}=F_n\cdot L_m \quad, \text{if n is odd and m is odd}\\ L_{n+m}-L_{n-m}=5\cdot F_n\cdot F_m, \text{if n is odd and m is even}\\ L_{n+m}-L_{n-m}=L_n\cdot L_m \quad, \text{if n is even and m is odd}\\ F_{n+m}-F_{n-m}=L_n\cdot F_m \quad, \text{if n is even and m is even}\end{cases},$$

respectively

$$(91)\begin{cases}L_{n+m}-L_{n-m}=L_n\cdot L_m \quad, \text{if n is odd and m is odd}\\ F_{n+m}-F_{n-m}=L_n\cdot F_m \quad, \text{if n is odd and m is even}\\ F_{n+m}-F_{n-m}=F_n\cdot L_m \quad, \text{if n is even and m is odd}\\ L_{n+m}-L_{n-m}=5\cdot F_n\cdot F_m, \text{if n is even and m is even}\end{cases}.$$

From the equalities (90) and (91), we deduce that:

$$(92)\begin{cases}L_{n+m}-L_{n-m}=5\cdot F_n\cdot F_m, \text{if m is even}\\ F_{n+m}-F_{n-m}=F_n\cdot L_m \quad, \text{if m is odd}\\ F_{n+m}-F_{n-m}=L_n\cdot F_m \quad, \text{if m is even}\\ L_{n+m}-L_{n-m}=L_n\cdot L_m \quad, \text{if m is odd}\end{cases},$$

that is:

$$(93)\begin{cases}L_{n+m}-L_{n-m}=[L_n\cdot L_m, 5\cdot F_n\cdot F_m]_m\\ F_{n+m}-F_{n-m}=[F_n\cdot L_m, L_n\cdot F_m]_m\end{cases}.$$

Therefore, the equalities (C.3) hold. For:

m=1,

from equalities (C.3), obtain that:

$$(94)\begin{cases}L_{n+1}-L_{n-1}=[L_n\cdot L_1, 5\cdot F_n\cdot F_1]_1\\ F_{n+1}-F_{n-1}=[F_n\cdot L_1, L_n\cdot F_1]_1\end{cases}.$$

Since:

$F_1=L_1=1$,

from equalities (94) şi (∗), obtain the equalities (C.3′):

$$\textbf{(95)}\begin{cases} L_{n+1}-L_{n-1}=L_n \\ F_{n+1}-F_{n-1}=F_n \end{cases},$$

which are even equalities (1) and (2).

Otherwise**:** From the equalities (9), (10) and (14), it follows that, for every m, n∈**N**:

$$\textbf{(96)}F_{n+m}\text{-}F_{n-m}=\frac{a^{n+m}-b^{n+m}}{a-b}-\frac{a^{n-m}-b^{n-m}}{a-b}=\frac{1}{a-b}\cdot\left[a^n\cdot a^m-b^n\cdot b^m-\frac{a^n}{a^m}-\frac{b^n}{b^m}\right]$$

$$=\frac{1}{a-b}\cdot\left[a^n\cdot a^m-b^n\cdot b^m-\frac{a^n\cdot b^m-a^m\cdot b^n}{(a\cdot b)^m}\right]$$

$$=\frac{1}{a-b}\cdot\left[a^n\cdot a^m-b^n\cdot b^m-\frac{a^n\cdot b^m-a^m\cdot b^n}{(-1)^m}\right]$$

$$=\begin{cases} \frac{1}{a-b}\cdot[a^n\cdot a^m-b^n\cdot b^m+a^n\cdot b^m-b^n\cdot a^m], \text{if m is odd} \\ \frac{1}{a-b}\cdot[a^n\cdot a^m-b^n\cdot b^m-a^n\cdot b^m+b^n\cdot a^m], \text{if m is even} \end{cases}$$

$$=\begin{cases} (a^m+b^m)\cdot\frac{a^n-b^n}{a-b}, \text{if m is odd} \\ \frac{a^m-b^m}{a-b}\cdot(a^n+b^n), \text{if m is even} \end{cases}$$

$$=\begin{cases} F_n\cdot L_m, \text{if m is odd} \\ L_n\cdot F_m, \text{if m is even} \end{cases}$$

$$=[F_n\cdot L_m, L_n\cdot F_m]_m;$$

respectively:

$$\textbf{(97)}L_{n+m}\text{-}L_{n-m}=a^{n+m}+b^{n+m}\text{-}a^{n-m}+b^{n-m}=a^n\cdot a^m+b^n\cdot b^m\text{-}\frac{a^n}{a^m}+\frac{b^n}{b^m}$$

$$=a^n\cdot a^m+b^n\cdot b^m\text{-}\frac{a^n\cdot b^m+a^m\cdot b^n}{(a\cdot b)^m}=a^n\cdot a^m+b^n\cdot b^m\text{-}\frac{a^n\cdot b^m+a^m\cdot b^n}{(-1)^m}$$

$$=\begin{cases} a^n\cdot a^m+b^n\cdot b^m+a^n\cdot b^m+b^n\cdot a^m, \text{if m is odd} \\ a^n\cdot a^m+b^n\cdot b^m-a^n\cdot b^m-b^n\cdot a^m, \text{if m is even} \end{cases}$$

$$=\begin{cases} (a^n+b^n)\cdot(a^m+b^m), \text{if m is odd} \\ (a^n-b^n)\cdot(a^m-b^m), \text{if m is even} \end{cases}$$

$$=\begin{cases} L_n\cdot L_m \quad, \text{if m is odd} \\ 5\cdot F_n\cdot F_m, \text{if m is even} \end{cases}$$

$$=[L_n\cdot L_m, 5\cdot F_n\cdot F_m]_m;$$

that is, we obtained again the equalities (C.3). Analogous, for:

m=1

obtain that:

(98) $$F_{n+1}-F_{n-1}=\frac{a^{n+1}-b^{n+1}}{a-b}-\frac{a^{n-1}-b^{n-1}}{a-b}=\frac{1}{a-b}\cdot\left[a^n\cdot a-b^n\cdot b-\frac{a^n}{a}-\frac{b^n}{b}\right]$$

$$=\frac{1}{a-b}\cdot\left[a^n\cdot a-b^n\cdot b-\frac{a^n\cdot b-a\cdot b^n}{a\cdot b}\right]=\frac{1}{a-b}\cdot\left[a^n\cdot a-b^n\cdot b-\frac{a^n\cdot b-a\cdot b^n}{-1}\right]$$

$$=\frac{1}{a-b}\cdot[a^n\cdot a-b^n\cdot b+a^n\cdot b-b^n\cdot a]=\frac{(a^n-b^n)\cdot(a+b)}{a-b}$$

$=F_n$;

respectively:

(99) $$L_{n+1}-L_{n-1}=a^{n+1}+b^{n+1}-a^{n-1}+b^{n-1}=a^n\cdot a+b^n\cdot b-\frac{a^n}{a}+\frac{b^n}{b}$$

$$=a^n\cdot a+b^n\cdot b-\frac{a^n\cdot b+a\cdot b^n}{a\cdot b}=a^n\cdot a+b^n\cdot b-\frac{a^n\cdot b+a\cdot b^n}{-1}$$

$$=a^n\cdot a+b^n\cdot b+a^n\cdot b+b^n\cdot a=(a^n+b^n)\cdot(a+b)$$

$=L_n$;

that is, we obtained again the equalities (C.3′). In equalities (98) and (99) we took into account the fact that, according to equality (5):

$a+b=1$.

24) + 25) from the equalities (C.2) and (C.3), we obtain that:

(100) $$F_{n+m}+F_{n-m}=\begin{cases}L_n\cdot F_m, \text{if } m \text{ is odd}\\ F_n\cdot L_m, \text{if } m \text{ is even}\end{cases},$$

(101) $$L_{n+m}+L_{n-m}=\begin{cases}5\cdot F_n\cdot F_m, \text{if } m \text{ is odd}\\ L_n\cdot L_m \quad, \text{if } m \text{ is even}\end{cases},$$

(102) $$F_{n+m}-F_{n-m}=\begin{cases}F_n\cdot L_m, \text{if } m \text{ is odd}\\ L_n\cdot F_m, \text{if } m \text{ is even}\end{cases},$$

(103) $$L_{n+m}-L_{n-m}=\begin{cases}L_n\cdot L_m \quad, \text{if } m \text{ is odd}\\ 5\cdot F_n\cdot F_m, \text{if } m \text{ is even}\end{cases}.$$

Suming the equalities (100) and (102), we obtain that:

(104) $$2\cdot F_{n+m}=\begin{cases}L_n\cdot F_m+F_n\cdot L_m, \text{if } m \text{ is odd}\\ F_n\cdot L_m+L_n\cdot F_m, \text{if } m \text{ is even}\end{cases},$$

that is:

(105) $2\cdot F_{n+m}=L_n\cdot F_m+F_n\cdot L_m$.

Suming the equalities (101) and (103), we obtain that:

(106) $$2\cdot L_{n+m}=\begin{cases}5\cdot F_n\cdot F_m+L_n\cdot L_m \,, \text{if } m \text{ is odd}\\ L_n\cdot L_m+5\cdot L_n\cdot F_m, \text{if } m \text{ is even}\end{cases},$$

that is:

(107) $2\cdot L_{n+m}=5\cdot F_n\cdot F_m+L_n\cdot L_m$.

The equalities (105) and (107) are exactly the equalities (C.4). In particular, for:

$m=1$,

from the equalities (105) and (107), we obtain that:

(108) $2\cdot F_{n+1}=L_n\cdot F_1+F_n\cdot L_1$.

respectively:

(109) $2\cdot L_{n+1}=5\cdot F_n\cdot F_1+L_n\cdot L_1$.

Since:

$L_1=F_1=1$,

from the equalities (108) and (109), we obtain that:

(110) $2\cdot F_{n+1}=L_n+F_n$.

respectively:

(111) $2\cdot L_{n+1}=5\cdot F_n+L_n$.

The equalities (110) and (111) are exactly the equalities (C.4′).

***Otherwise*:** According to the equalities (9), (10) and (14), it follows that, for every m, n$\in$**N**:

(112)
$$\begin{aligned}L_n\cdot L_m+5\cdot F_n\cdot F_m&=(a^n+b^n)\cdot(a^m+b^m)+(a^n-b^n)\cdot(a^m-b^m)\\&=a^{n+m}+a^n\cdot b^m+b^n\cdot a^m+b^{n+m}+a^{n+m}-a^n\cdot b^m-b^n\cdot a^m+b^{n+m}\\&=2\cdot(a^{n+m}+b^{n+m})\\&=2\cdot L_{n+m};\end{aligned}$$

respectively:

(113)
$$\begin{aligned}F_n\cdot L_m+L_n\cdot F_m&=\frac{1}{a-b}\cdot[(a^n-b^n)\cdot(a^m+b^m)+(a^n+b^n)\cdot(a^m-b^m)]\\&=\frac{1}{a-b}\cdot(a^{n+m}+a^n\cdot b^m-b^n\cdot a^m-b^{n+m}+a^{n+m}-a^n\cdot b^m+b^n\cdot a^m-b^{n+m})\\&=2\cdot\frac{1}{a-b}\cdot(a^{n+m}-b^{n+m})\\&=2\cdot F_{n+m}.\end{aligned}$$

The equalities (112) and (113) are exactly the equalities (C.4). On the other hand,

(114)
$$\begin{aligned}L_n+5\cdot F_n&=a^n+b^n+(a^n-b^n)\cdot(a-b)=a^n\cdot(1+a-b)+b^n\cdot(1-a+b)\\&=a^n\cdot(a+b+a-b)+b^n\cdot(a+b-a+b)=2\cdot(a^{n+1}+b^{n+1})\\&=2\cdot L_{n+1};\end{aligned}$$

respectively:

(115)
$$\begin{aligned}F_n+L_n&=\frac{1}{a-b}\cdot(a^n-b^n)+(a^n+b^n)=\frac{1}{a-b}\cdot[(a^n-b^n)+(a^n+b^n)\cdot(a-b)]\\&=\frac{1}{a-b}\cdot(a^n-b^n+a^{n+1}-a^n\cdot b+b^n\cdot a-b^{n+1})=\frac{1}{a-b}\cdot[a^n\cdot(1+a-b)-b^n\cdot(1-a+b)]\end{aligned}$$

$$=\frac{1}{a-b}\cdot[a^n\cdot(a+b+a-b)-b^n\cdot(a+b-a+b)]=2\cdot\frac{1}{a-b}\cdot(a^{n+1}-b^{n+1})$$

$$=2\cdot F_{n+1}.$$

The equalities (114) and (115) are exactly the equalities (C.4′).

26) + 27) Subtracting the equalities (100) and (102), we obtain that:

(116) $2\cdot F_{n-m}=\begin{cases} L_n\cdot F_m - F_n\cdot L_m, \text{if } m \text{ is odd} \\ F_n\cdot L_m - L_n\cdot F_m, \text{if } m \text{ is even}\end{cases}$,

that is:

(117) $2\cdot F_{n-m}=(-1)^{m-1}\cdot(L_n\cdot F_m-F_n\cdot L_m)$.

Subtracting the equalities (101) and (103), we obtain that:

(118) $2\cdot L_{n-m}=\begin{cases} 5\cdot F_n\cdot F_m - L_n\cdot L_m, \text{if } m \text{ is odd} \\ L_n\cdot L_m - 5\cdot L_n\cdot F_m, \text{if } m \text{ is even}\end{cases}$,

that is:

(119) $2\cdot L_{n+m}=(-1)^{m-1}\cdot(5\cdot F_n\cdot F_m-L_n\cdot L_m)$.

The equalities (117) and (119) are exactly the equalities (C.5). In particular, for:

$m=1$,

from the equalities (117) şi (119), we obtain that:

(120) $2\cdot F_{n-1}=L_n\cdot F_1-F_n\cdot L_1$.

respectively:

(121) $2\cdot L_{n-1}=5\cdot F_n\cdot F_1-L_n\cdot L_1$.

Since:

$L_1=F_1=1$,

from the equalities (120) and (121), we obtain that:

(122) $2\cdot F_{n+1}=L_n-F_n$.

respectively:

(123) $2\cdot L_{n+1}=5\cdot F_n-L_n$.

The equalities (122) and (123) are exactly the equalities (C.5′).

***Otherwise*:** According to the equalities (9), (10) and (14), it follows that, for every m, n$\in$**N**:

(124) $5\cdot F_n\cdot F_m- L_n\cdot L_m=(a^n-b^n)\cdot(a^m-b^m)-(a^n+b^n)\cdot(a^m+b^m)$

$$=a^{n+m}-a^n\cdot b^m-b^n\cdot a^m+b^{n+m}-a^{n+m}-a^n\cdot b^m-b^n\cdot a^m-b^{n+m}$$

$$=-2\cdot(a^n\cdot b^m+b^n\cdot a^m)=-2\cdot(-1)^m\cdot\frac{a^n\cdot b^m+b^n\cdot a^m}{(-1)^m}$$

$$=2\cdot(-1)^{m-1}\cdot\frac{a^n\cdot b^m+b^n\cdot a^m}{a^m\cdot b^m}=2\cdot(-1)^{m-1}\cdot(a^{n-m}+b^{n-m})$$

$$=2\cdot(-1)^{m-1}\cdot L_{n-m};$$

respectively:

(125) $L_n \cdot F_m - F_n \cdot L_m = \frac{1}{a-b} \cdot [(a^n+b^n)\cdot(a^m-b^m)-(a^n-b^n)\cdot(a^m+b^m)]$

$$= \frac{1}{a-b} \cdot (a^{n+m}-a^n\cdot b^m+b^n\cdot a^m-b^{n+m}-a^{n+m}-a^n\cdot b^m+b^n\cdot a^m+b^{n+m})$$

$$=-2\cdot \frac{1}{a-b} \cdot (a^n\cdot b^m-b^n\cdot a^m)=-2\cdot \frac{1}{a-b} \cdot (-1)^m \cdot \frac{a^n\cdot b^m-b^n\cdot a^m}{(-1)^m}$$

$$=2\cdot \frac{1}{a-b} \cdot (-1)^{m-1} \cdot \frac{a^n\cdot b^m-b^n\cdot a^m}{a^m\cdot b^m} =2\cdot \frac{1}{a-b} \cdot (-1)^{m-1}\cdot(a^{n-m}-b^{n-m})$$

$$=2\cdot(-1)^{m-1}\cdot F_{n-m}.$$

The equalities (124) and (125) are exactly the equalities (C.5). On the other hand,

(126) $L_n-F_n=a^n+b^n-(a^n-b^n)\cdot \frac{1}{a-b} = \frac{(a^n+b^n)\cdot(a-b)-a^n+b^n}{a-b}$

$$= \frac{a^{n+1}-a^n\cdot b+b^n\cdot a-b^{n+1}-a^n+b^n}{a-b} = \frac{a^n\cdot(a-b-1)+b^n\cdot(a-b+1)}{a-b}$$

$$= \frac{a^n\cdot(a-b-a-b)+b^n\cdot(a-b+a+b)}{a-b} = \frac{-a^n\cdot b+b^n\cdot a}{a-b}$$

$$=2\cdot \frac{-a^n\cdot b+b^n\cdot a}{a-b} =2\cdot \frac{a^n\cdot b-b^n\cdot a}{(-1)\cdot(a-b)} =2\cdot \frac{a^n\cdot b-b^n\cdot a}{a\cdot b\cdot(a-b)} =2\cdot \frac{a^{n-1}-b^{n-1}}{a-b}$$

$$=2\cdot F_{n-1};$$

respectively:

(127) $5\cdot F_n-L_n=(a-b)\cdot(a^n-b^n)-(a^n+b^n)=a^{n+1}-b^n\cdot a-a^n\cdot b+b^{n+1}-a^n-b^n$

$$=a^n\cdot(a-b-1)+b^n\cdot(-a+b-1)=a^n\cdot(a-b-a-b)+b^n\cdot(-a+b-a-b)$$

$$=-2\cdot(a^n\cdot b+b^n\cdot a)= \frac{1}{a\cdot b} \cdot 2\cdot(a^n\cdot b+b^n\cdot a)=2\cdot(a^{n-1}+b^{n-1})$$

$$=2\cdot L_{n-1}.$$

The equalities (126) and (127) are exactly the equalities (C.5′).

28) + 29) + 30) By multiplying, member by member, the equalities (3.2) and (3.3) and taking into account the equality (3.1), we obtain that, for every x, y$\in\mathbf{R}$:

(128) $sh(x+y)\cdot sh(x-y)=(shx\cdot chy+chx\cdot shy)\cdot(shx\cdot chy-chx\cdot shy)=sh^2x\cdot ch^2y-ch^2x\cdot sh^2y$

$$=sh^2x\cdot(1+sh^2y)-(1+sh^2x)\cdot sh^2y=sh^2x+sh^2x\cdot sh^2y-sh^2y-sh^2x\cdot sh^2y$$

$$=sh^2x-sh^2y.$$

On the other hand, by multiplying, member by member, the equalities (3.4) and (3.5) and taking into account the same equality (3.1), we obtain that, for every x, y$\in\mathbf{R}$:

(129) $ch(x+y)\cdot ch(x-y)=(chx\cdot chy+shx\cdot shy)\cdot(chx\cdot chy-shx\cdot shy)=ch^2x\cdot ch^2y-sh^2x\cdot sh^2y$

$$=ch^2x\cdot ch^2y-(ch^2x-1)\cdot sh^2y=ch^2x\cdot(1+sh^2y)-ch^2x\cdot sh^2y+sh^2y$$

$$=ch^2x+ch^2x\cdot sh^2y-ch^2x\cdot sh^2y+sh^2y$$

$$=ch^2x+sh^2y.$$

Now, from the equalities (128), (A.1), (A.3) and (A.5), we obtain that, for every m, n$\in$**N**:

(130) $\frac{1}{2}\cdot[L_{n+m}, \sqrt{5}\cdot F_{n+m}]_{n+m}\cdot\frac{1}{2}\cdot[L_{n-m}, \sqrt{5}\cdot F_{n-m}]_{n-m}=\frac{1}{4}\cdot([L_n, \sqrt{5}\cdot F_n]_n)^2-\frac{1}{4}\cdot([L_m, \sqrt{5}\cdot F_m]_m)^2$,

that is:

(131) $[L_{n+m}, \sqrt{5}\cdot F_{n+m}]_{n+m}\cdot[L_{n-m}, \sqrt{5}\cdot F_{n-m}]_{n-m}=([L_n, \sqrt{5}\cdot F_n]_n)^2-([L_m, \sqrt{5}\cdot F_m]_m)^2$.

So:

$$\textbf{(132)}\quad \begin{cases} \sqrt{5}\cdot F_{n+m}\cdot\sqrt{5}\cdot F_{n-m}=L_n^2-L_m^2 & \text{, if n is odd and m is odd} \\ L_{n+m}\cdot L_{n-m}=L_n^2-5\cdot F_m^2 & \text{, if n is odd and m is even} \\ L_{n+m}\cdot L_{n-m}=5\cdot F_n^2-L_m^2 & \text{, if n is even and m is odd} \\ \sqrt{5}\cdot F_{n+m}\cdot\sqrt{5}\cdot F_{n-m}=5\cdot F_n^2-5\cdot F_m^2 & \text{, if n is even and m is even} \end{cases},$$

that is:

$$\textbf{(133)}\quad \begin{cases} 5\cdot F_{n+m}\cdot F_{n-m}=L_n^2-L_m^2 & \text{, if n is odd and m is odd} \\ L_{n+m}\cdot L_{n-m}=L_n^2-5\cdot F_m^2 & \text{, if n is odd and m is even} \\ L_{n+m}\cdot L_{n-m}=5\cdot F_n^2-L_m^2 & \text{, if n is even and m is odd} \\ F_{n+m}\cdot F_{n-m}=F_n^2-F_m^2 & \text{, if n is even and m is even} \end{cases}.$$

On the other hand, from the equalities (129), (A.1), (A.2), (A.4) and (A.6), we obtain that, for every m, n$\in$**N**:

(134) $\frac{1}{2}\cdot[\sqrt{5}\cdot F_{n+m},L_{n+m}]_{n+m}\cdot\frac{1}{2}\cdot[\sqrt{5}\cdot F_{n-m},L_{n-m}]_{n-m}=\frac{1}{4}\cdot([\sqrt{5}\cdot F_n,L_n]_n)^2+\frac{1}{4}\cdot([L_m, \sqrt{5}\cdot F_m]_m)^2$,

that is:

(135) $[\sqrt{5}\cdot F_{n+m},L_{n+m}]_{n+m}\cdot[\sqrt{5}\cdot F_{n-m},L_{n-m}]_{n-m}=([\sqrt{5}\cdot F_n,L_n]_n)^2+([L_m, \sqrt{5}\cdot F_m]_m)^2$.

So:

$$\textbf{(136)}\quad \begin{cases} L_{n+m}\cdot L_{n-m}=5\cdot F_n^2+L_m^2 & \text{, if n is odd and m is odd} \\ \sqrt{5}\cdot F_{n+m}\cdot\sqrt{5}\cdot F_{n-m}=5\cdot F_n^2+5\cdot F_m^2 & \text{, if n is odd and m is even} \\ \sqrt{5}\cdot F_{n+m}\cdot\sqrt{5}\cdot F_{n-m}=L_n^2+L_m^2 & \text{, if n is even and m is odd} \\ L_{n+m}\cdot L_{n-m}=L_n^2+5\cdot F_m^2 & \text{, if n is even and m is even} \end{cases},$$

that is:

$$\textbf{(137)}\quad \begin{cases} L_{n+m}\cdot L_{n-m}=5\cdot F_n^2+L_m^2 & \text{, if n is odd and m is odd} \\ F_{n+m}\cdot F_{n-m}=F_n^2+F_m^2 & \text{, if n is odd and m is even} \\ 5\cdot F_{n+m}\cdot F_{n-m}=L_n^2+L_m^2 & \text{, if n is even and m is odd} \\ L_{n+m}\cdot L_{n-m}=L_n^2+5\cdot F_m^2 & \text{, if n is even and m is even} \end{cases}.$$

From the equalities (133) and (137), we obtain that, obţinem că, for every n, m$\in$**N**,

(138) $$\begin{cases} L_{n+m}\cdot L_{n-m}-L_n^2=5\cdot F_n^2+L_m^2-L_n^2 & \text{, if n is odd and m is odd} \\ L_{n+m}\cdot L_{n-m}-L_n^2=5\cdot F_n^2-L_m^2-L_n^2 & \text{, if n is even and m is odd} \\ L_{n+m}\cdot L_{n-m}-L_n^2=5\cdot F_m^2 & \text{, if n is even and m is even} \\ L_{n+m}\cdot L_{n-m}-L_n^2=-5\cdot F_m^2 & \text{, if n is odd and m is even} \\ F_{n+m}\cdot F_{n-m}-F_n^2=F_m^2 & \text{, if n is odd and m is even} \\ F_{n+m}\cdot F_{n-m}-F_n^2=-F_m^2 & \text{, if n is even and m is even} \\ 5\cdot F_{n+m}\cdot F_{n-m}-5\cdot F_n^2=L_n^2+L_m^2-5\cdot F_n^2 & \text{, if n is even and m is odd} \\ 5\cdot F_{n+m}\cdot F_{n-m}-5\cdot F_n^2=L_n^2-L_m^2-5\cdot F_n^2 & \text{, if n is even and m is odd} \end{cases};$$

that is, according to the equalities (10) and (14):

(139) $$\begin{cases} F_{n+m}\cdot F_{n-m}-F_n^2=(-1)^{n+m+1}\cdot F_m^2 \\ L_{n+m}\cdot L_{n-m}-L_n^2=(-1)^{n+m}\cdot L_m^2-4\cdot(-1)^n \end{cases}.$$

From the equalities (139), for:

$$m=1$$

obtain that:

(140) $$\begin{cases} L_{n+1}\cdot L_{n-1}=L_n^2+5\cdot(-1)^{n+1} \\ F_{n+1}\cdot F_{n-1}=F_n^2+(-1)^n \end{cases};$$

because:

$$F_1=L_1=1,$$

and, for:

$$m=2,$$

from the same equalities (139), we obtain that:

(141) $$\begin{cases} L_{n+2}\cdot L_{n-2}=L_n^2+5\cdot(-1)^n \\ F_{n+2}\cdot F_{n-2}=F_n^2+(-1)^{n+1} \end{cases}.$$

The equalities (140) and (141) show that the equalities (C.6′) and (C.6″) also hold.

***Otherwise*:** According to the equalities (10) and (14), for every m, n∈**N**,

(142) $$\begin{aligned} L_{n+m}\cdot L_{n-m}-L_n^2 &= (a^{n+m}+b^{n+m})\cdot(a^{n-m}+b^{n-m})-(a^n+b^n)^2 \\ &= a^{2n}+a^{n+m}\cdot b^{n-m}+a^{n-m}\cdot b^{n+m}+b^{2n}-a^{2n}-2\cdot a^n\cdot b^n-b^{2n} \\ &= a^{n+m}\cdot b^{n-m}+a^{n-m}\cdot b^{n+m}-2\cdot a^n\cdot b^n \\ &= (-1)^n\cdot\left(\frac{a^m}{b^m}-2+\frac{b^m}{a^m}\right)=(-1)^n\cdot\left[\frac{(a^m+b^m)^2}{(a\cdot b)^m}-4\right] \\ &= (-1)^{n+m}\cdot(a^m+b^m)^2-4\cdot(-1)^n, \end{aligned}$$

and

(143) $F_{n+m}\cdot F_{n-m}-F_n^2=\frac{a^{n+m}-b^{n+m}}{a-b}\cdot\frac{a^{n-m}-b^{n-m}}{a-b}-\left(\frac{a^n-b^n}{a-b}\right)^2$

$$=\frac{a^{2n}-a^{n+m}\cdot b^{n-m}-a^{n-m}\cdot b^{n+m}+b^{2n}-a^{2n}+2\cdot a^n\cdot b^n-b^{2n}}{(a-b)^2}$$

$$=\frac{-a^{n+m}\cdot b^{n-m}-a^{n-m}\cdot b^{n+m}+2\cdot a^n\cdot b^n}{(a-b)^2}$$

$$=(-1)^{n+1}\cdot\frac{1}{(a-b)^2}\cdot\left(\frac{a^m}{b^m}-2+\frac{b^m}{a^m}\right)=(-1)^{n+1}\cdot\frac{1}{(a-b)^2}\cdot\frac{(a^m-b^m)^2}{(a\cdot b)^m}$$

$$=(-1)^{n+m+1}\cdot F_m^2.$$

The equalities (142) and (143) are exactly the equalities (C.6). On the other hand, according to the same equalities (10) and (14), for every $n\in\mathbf{N}$,

(144) $L_{n+1}\cdot L_{n-1}-L_n^2=(a^{n+1}+b^{n+1})\cdot(a^{n-1}+b^{n-1})-(a^n+b^n)^2$

$$=a^{2n}+a^{n+1}\cdot b^{n-1}+a^{n-1}\cdot b^{n+1}+b^{2n}-a^{2n}-2\cdot a^n\cdot b^n-b^{2n}$$

$$=a^{n+1}\cdot b^{n-1}+a^{n-1}\cdot b^{n+1}-2\cdot a^n\cdot b^n=(-1)^n\cdot\left(\frac{a}{b}-2+\frac{b}{a}\right)$$

$$=(-1)^n\cdot\left[\frac{(a+b)^2}{a\cdot b}-4\right]=(-1)^{n+1}\cdot(a+b)^2-4\cdot(-1)^n=(-1)^{n+1}-4\cdot(-1)^n$$

$$=5\cdot(-1)^{n+1}$$

and

(145) $F_{n+1}\cdot F_{n-1}-F_n^2=\frac{a^{n+1}-b^{n+1}}{a-b}\cdot\frac{a^{n-1}-b^{n-1}}{a-b}-\left(\frac{a^n-b^n}{a-b}\right)^2$

$$=\frac{a^{2n}-a^{n+1}\cdot b^{n-1}-a^{n-1}\cdot b^{n+1}+b^{2n}-a^{2n}+2\cdot a^n\cdot b^n-b^{2n}}{(a-b)^2}$$

$$=\frac{-a^{n+1}\cdot b^{n-1}-a^{n-1}\cdot b^{n+1}+2\cdot a^n\cdot b^n}{(a-b)^2}=(-1)^{n+1}\cdot\frac{1}{(a-b)^2}\cdot\left(\frac{a}{b}-2+\frac{b}{a}\right)$$

$$=(-1)^{n+1}\cdot\frac{1}{(a-b)^2}\cdot\frac{(a-b)^2}{a\cdot b}=(-1)^n$$

$$=(-1)^{n+2}.$$

The equalities (144) and (145) show that we obtained, agai, the equalities (C.6′). Finally, also according to the same equalities (10) and (14), for every $n\in\mathbf{N}$,

(146) $L_{n+2}\cdot L_{n-2}-L_n^2=(a^{n+2}+b^{n+2})\cdot(a^{n-2}+b^{n-2})-(a^n+b^n)^2=a^{2n}+a^{n+2}\cdot b^{n-1}+a^{n-2}\cdot b^{n+1}+b^{2n}-a^{2n}-2\cdot a^n\cdot b^n-b^{2n}$

$$=a^{n+2}\cdot b^{n-2}+a^{n-2}\cdot b^{n+2}-2\cdot a^n\cdot b^n=(-1)^n\cdot\left(\frac{a^2}{b^2}-2+\frac{b^2}{a^2}\right)=(-1)^n\cdot\left[\frac{(a^2+b^2)^2}{(a\cdot b)^2}-4\right]$$

$$=(-1)^{n+2}\cdot(a^2+b^2)^2-4\cdot(-1)^n=(-1)^n\cdot L_2^2-4\cdot(-1)^n=(-1)^n\cdot 9-4\cdot(-1)^n$$

$=5\cdot(-1)^n$

and

(147) $$F_{n+1}\cdot F_{n-1}-F_n^2=\frac{a^{n+2}-b^{n+2}}{a-b}\cdot\frac{a^{n-2}-b^{n-2}}{a-b}-\left(\frac{a^n-b^n}{a-b}\right)^2$$

$$=\frac{a^{2n}-a^{n+2}\cdot b^{n-2}-a^{n-2}\cdot b^{n+2}+b^{2n}-a^{2n}+2\cdot a^n\cdot b^n-b^{2n}}{(a-b)^2}$$

$$=\frac{-a^{n+2}\cdot b^{n-2}-a^{n-2}\cdot b^{n+2}+2\cdot a^n\cdot b^n}{(a-b)^2}=(-1)^{n+1}\cdot\frac{1}{(a-b)^2}\cdot\left(\frac{a^2}{b^2}-2+\frac{b^2}{a^2}\right)$$

$$=(-1)^{n+1}\cdot\frac{1}{(a-b)^2}\cdot\frac{(a^2-b^2)^2}{(a\cdot b)^2}=(-1)^{n+1}\cdot(a+b)^2$$

$$=(-1)^{n+1}.$$

The equalities (146) and (147) show that we obtained again the equalities (C.6″).

4. Conclusions

Here, then, that these functions, through their properties, have interesting applications in Algebra, see also (Vălcan, (1), 2019). Of course, the reader who is attentive and interested in these issues, using the results of the works in the bibliographic list, can obtain other applications of hyperbolic functions, not only in Algebra, but also in Geometry or Mathematical Analysis.

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